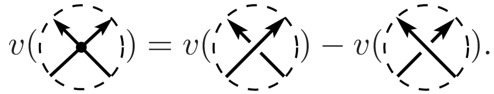
On the $\mathfrak{sl}_2$ Weight System and Intersection Graphs

Huan Vo
(Joint work with Dror Bar-Natan)

University of Toronto

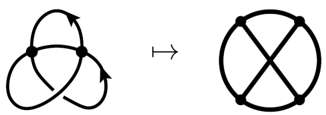
April 30, 2016

- Let $\mathcal{K}$ be the set of (isotopy classes of) knots in S^3. Given a knot invariant $v\colon \mathcal{K} \to \mathbb{Q}$ we can extend v to *singular knots* by

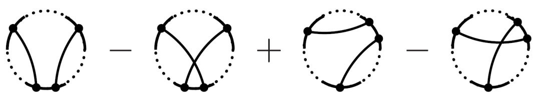


- We call an invariant v **finite type (or Vassiliev) of degree** d if v vanishes for any knot with $d+1$ singular points.

- If v is a finite-type invariant of degree d, then the value of v on a knot with d singular points only depends on the position of the singular points and not on how the knot is embedded in S^3, i.e. a **chord diagram**



- Let $\mathcal{A}$ be the vector space over $\mathbb{Q}$ spanned by all chord diagrams, modulo the **4T relation**:



Lemma

Every finite-type invariant $v\colon \mathcal{K} \to \mathbb{Q}$ of degree d induces a linear functional

$$W_v\colon \mathcal{A}^d \to \mathbb{Q},$$

where $\mathcal{A}^d$ denotes the subspace of $\mathcal{A}$ spanned by all chord diagrams of degree d (=number of chords).

We call a linear functional $W\colon \mathcal{A} \to \mathbb{Q}$ a **weight system**.

Theorem (Fundamental Theorem of Vassiliev Invariants [BN95])

Every weight system $W\colon \mathcal{A}^d \to \mathbb{Q}$ gives rise to a finite-type invariant of degree d. More precisely, there exists a knot invariant

$$Z\colon \mathcal{K} \to \mathcal{A}$$

so that $v = W \circ Z\colon \mathcal{K} \to \mathbb{Q}$ is a finite-type invariant of degree d and $W_v = W$.

Here the invariant Z is obtained from the celebrated **Kontsevich Integral**

$$I(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi\sqrt{-1})^m} \times \int\limits_{\substack{t_{\min}<t_m<\cdots<t_1<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}.$$

- Given a semisimple Lie algebra $\mathfrak{g}$ and a representation R thereof, we can construct a weight system (see [CDM12]):

$$W_{\mathfrak{g},R}\colon \mathcal{A} \to Z(\mathcal{U}\mathfrak{g}) \xrightarrow{\mathrm{tr}_R} \mathbb{Q}.$$

- When $\mathfrak{g} = \mathfrak{sl}_2$, the center of $\mathcal{U}\mathfrak{sl}_2$ is isomorphic to the algebra of polynomials in c, the Casimir element of $\mathfrak{sl}_2$.
- So a pair $(\mathfrak{g}, R)$ gives us a knot invariant. The relationship with quantum invariants is as follows.

Theorem ([Oht02])

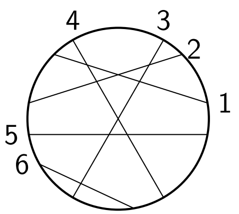
Let $Q_{\mathfrak{g},R}\colon \mathcal{K} \to \mathbb{C}$ be the quantum invariant obtained from $\mathcal{U}_q(\mathfrak{g})$ (a.k.a. a quantum group). By letting $q = e^h$ we have

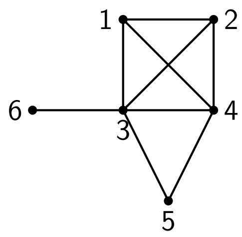
$$Q_{\mathfrak{g},R}(K)|_{q=e^h} = \sum_{d=0}^{\infty} a_d(K) h^d.$$

Then each a_d is a finite-type invariant of degree d and moreover $W_{a_d} = W_{\mathfrak{g},R}|_{\mathcal{A}^d}$.

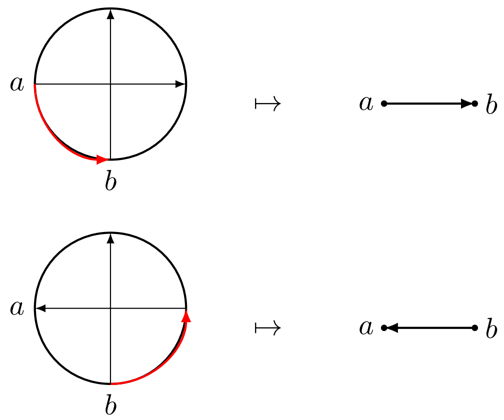
So in particular finite-type invariants contain quantum invariants.

To every chord diagram D, we can associate with it a graph $\Gamma(D)$, called the **intersection graph**:

$D =$ 

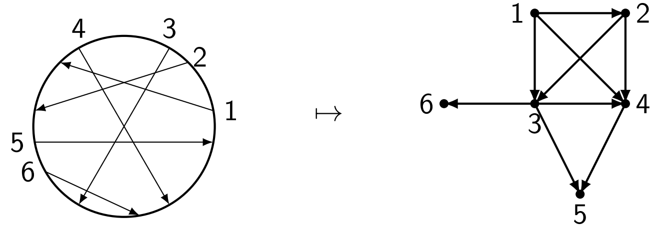
$\mapsto$ $\Gamma(D) =$ 

If we equip each chord with an orientation, then we obtain a directed graph as follows:



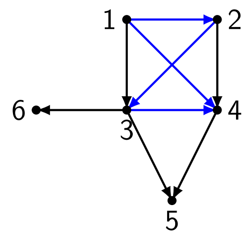
(Recall that we always orient the skeleton counter-clockwise.)

So for example

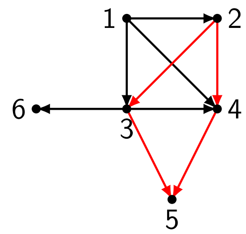


- Given a chord diagram D, by orient the chords of D arbitrarily, we obtain a directed graph $\overline{\Gamma(D)}$.
- By a **circuit** we mean a closed path with no repeated vertices (except the first and the last vertices).
- For a circuit s of even length, we can associate with it a **sign** defined as follows:

$$\mathrm{sign}(s) := (-1)^{\#\text{ of edges in } s \text{ with the opposite orientation}}.$$



For example, the circuit $(1, 2, 3, 4)$ has sign -1.



The circuit $(2, 3, 5, 4)$ has sign $+1$.

The sign of a circuit does not depend on how we orient the chord diagram since

- changing the orientation of a single chord a will reverse the orientations of all the edges incident to a,
- each circuit contains exactly zero or two edges incident to a.

Therefore the following definition makes sense.

Definition

Given a chord diagram D and an interger $m > 1$, let

$$R_m(D) := \sum_s \mathrm{sign}(s),$$

where the sum is over all circuits s of length $2m$ in $\overline{\Gamma(D)}$, the oriented intersection graph of D.

Theorem ([KLMR14])

The function R_m on chord diagrams is indeed a weight system, that is, it satisfies the 4T relation.

Now to describe the relationship between R_m and the universal weight system coming from $\mathfrak{sl}_2$, we need to introduce a projection map defined as follows.

Theorem ([Lan97])

Given a chord diagram D with n chords, consider the following map

$$\pi_n(D) := D - \frac{1}{2} \sum_{V=V_1 \sqcup V_2} D_1 \cdot D_2 + \frac{1}{3} \sum_{V=V_1 \sqcup V_2 \sqcup V_3} D_1 \cdot D_2 \cdot D_3 - \cdots + \frac{(-1)^{n-1}}{n} \sum_{V=V_1 \sqcup V_2 \sqcup \cdots \sqcup V_n} D_1 \cdot D_2 \cdots D_n,$$

where sums are taken over all ordered disjoint partitions of V into non-empty subsets and D_i denotes D with only chords from V_i. Then $\pi_n(D)$ is primitive. (Here V is the set of chords of D.)

Our main theorem is

Theorem ([BNV15])

Let D be a chord diagram with $2m$ chords $(m > 1)$, and $w_{\mathfrak{sl}_2,2}$ be the weight system coming from the Lie algebra $\mathfrak{sl}_2$ equipped with the invariant form $2\langle \cdot, \cdot \rangle$. Then

$$w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) = 2R_m(D)c_2^m + \text{ terms of degree less than } m \text{ in } c_2.$$

Here c_2 is the Casimir element associated with $(\mathfrak{sl}_2, 2\langle \cdot, \cdot \rangle)$ and

$$\langle x, y \rangle = \mathrm{Tr}(xy), \quad x, y \in \mathfrak{sl}_2,$$

where we think of x and y as usual 2×2 matrices.

This formula first appeared in [KLMR14].

- Let $J^k(q)$ be the colored Jones polynomial associated with the irreducible representation of $\mathfrak{sl}_2$ with highest weight $k-1$.
- Set $q = e^h$, express $J^k(q)$ as power series in h:

$$J^k = \sum_{n=0}^{\infty} J_n^k h^n.$$

- It is known that J_n^k is a polynomial in k of degree at most $n+1$ with no constant term. Therefore we can write

$$\frac{J^k}{k} = \sum_{n=0}^{\infty} \left(\sum_{0 \le j \le n} b_{n,j} k^j \right) h^n,$$

Definition

We denote the highest order part of the colored Jones polynomial by

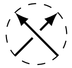
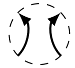
$$JJ :\,= \sum_{n=0}^{\infty} b_{n,n} h^n.$$

Now let $C(t)$ be the Conway polynomial. It is defined via the skein relation:

(i) $C(\text{unknot}) = 1$

(ii) $C(L_+) - C(L_-) = tC(L_0)$,

where

$L_+ =$  $L_- =$  $L_0 =$ 

Definition

The Alexander-Conway power series is given by

$$\widetilde{C}(h) :\,= \frac{h}{e^{h/2} - e^{-h/2}} C(e^{h/2} - e^{-h/2}) = \sum_{n=0}^{\infty} c_n h^n.$$

Theorem (MMR Conjecture)

We have

$$JJ(h)(K) \cdot \widetilde{C}(h)(K) = 1$$

for any knot K.

For our purpose, we reformulate the MMR conjecture in the language of weight systems following Bar-Natan and Garoufalidis.

Lemma ([BNG96])

Let

$$W_{JJ} := \sum_{n=0}^{\infty} W_n(b_{n,n}) \text{ and } W_C := \sum_{n=0}^{\infty} W_n(c_n)$$

be the weight systems of JJ and $\widetilde{C}$ respectively. Then the MMR conjecture is equivalent to

$$W_{JJ} \cdot W_C = \mathbf{1}.$$

Here $\mathbf{1}$ denotes the weight system that takes value 1 on the empty chord diagram and 0 otherwise.

- Recall also that the product of two weight systems is given by

$$(W_{JJ} \cdot W_C)(D) = \sum_{V = V_1 \sqcup V_2} W_{JJ}(D_1) \cdot W_C(D_2),$$

- In particular, when D is primitive, we have

$$0 = (W_{JJ} \cdot W_C)(D) = W_{JJ}(D) + W_C(D).$$

- Thus we obtain

Lemma

If D is a chord diagram of degree $2m$, then

$$W_{JJ}(\pi_{2m}(D)) = -W_C(\pi_{2m}(D)).$$

Now to make connection with $R_m(D)$, we have the following lemma

Lemma ([BNG96])

Given a chord diagram D of degree $2m$, we have

$$-W_C(\pi_{2m}(D)) = 2R_m(D).$$

- To summarize, we have a chain of equalities

$$2R_m(D) = -W_C(\pi_{2m}(D)) = W_{JJ}(\pi_{2m}(D)).$$

- Therefore,

$$\frac{J^k_{2m}(\pi_{2m}(D))}{k} = 2R_m(D)k^{2m} + \text{ terms of degree less than } 2m \text{ in } k.$$

- Recall that the symbol of J^k is the $\mathfrak{sl}_2$ weight system, with the Casimir element $c = \left(\frac{k^2-1}{2}\right) I_k$. Thus

$$w_{\mathfrak{sl}_2}(\pi_{2m}(D)) = 2^{m+1} R_m(D) c^m + \text{ terms of degree less than } m \text{ in } c.$$

- Finally, we do a change of variable

$$\begin{aligned} w_{\mathfrak{sl}_2,2}(\pi_{2m}(D)) &= \frac{1}{2^{2m}}\, w_{\mathfrak{sl}_2}(\pi_{2m}(D))|_{c=2c_2} \\ &= 2R_m(D)c_2^m + \text{ terms of degree less than } m \text{ in } c_2, \end{aligned}$$

which completes the proof.

References I

[BN95] Dror Bar-Natan. *On the Vassiliev Knot Invariants*. *Topology*, 34:423–472, 1995.

[BNG96] D. Bar-Natan and S. Garoufalidis. *On the Melvin-Morton-Rozansky Conjecture*. *Inventiones Mathematicae*, 125:103–133, 1996.

[BNV15] Dror Bar-Natan and Huan Vo. *Proof of a Conjecture of Kulakova et al. Related to the $\mathfrak{sl}_2$ Weight System*. *European J. Combin.*, 45:65–70, 2015.

[CDM12] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

[KLMR14] E. Kulakova, S. Lando, T. Mukhutdinova, and G. Rybnikov. *On a Weight System Conjecturally Related to $\mathfrak{sl}_2$*. *European J. Combin.*, 41:266–277, 2014.

References II

[Lan97] S. K. Lando. *On Primitive Elements in the Bialgebra of Chord Diagrams*. *Amer. Math. Soc. Transl. Ser. 2, AMS, Providence RI*, 180:167–174, 1997.

[Oht02] Tomotada Ohtsuki. *Quantum Invariants: a Study of Knots, 3-Manifolds and Their Sets*. World Scientific, 2002.

THANK YOU