On Meta-monoids and the Alexander Polynomial

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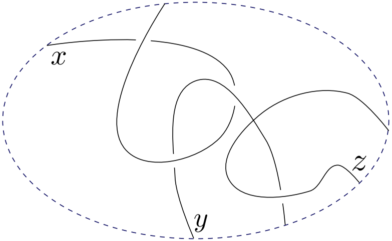
Agenda

- w-Tangles
- Meta-monoids
- Γ-calculus
- Ribbon knots

w-Tangles

Definition

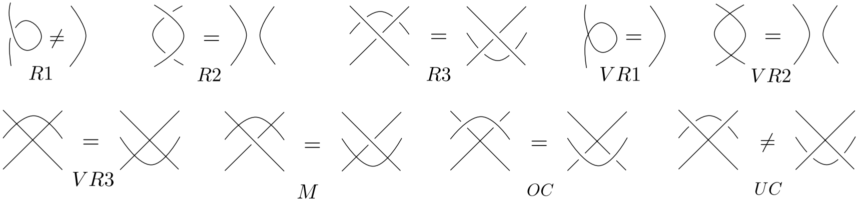
A **w-tangle diagram** is a finite collection of oriented arcs (or *components*) smoothly drawn on a plane, with finitely many intersections, divided into *virtual crossings* , *positive crossings* , and *negative crossings* ; and regarded up to planar isotopy. We also require distinct components to be labeled by distinct labels from some set of labels X. Note that we do NOT allow closed components.



w-Tangles

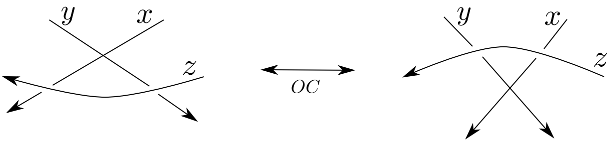
Definition

A **w-tangle** is an equivalence class of w-tangle diagrams, modulo the equivalence generated by the Reidemeister 2 and 3 moves ($R2$, and $R3$), the virtual Reidemeister 1 through 3 moves ($VR1$, $VR2$, $VR3$), the mixed relations (M), and the overcrossings commute relations (OC).



w-Tangles

Let us describe the OC relations in a bit more details.



It says that from the perspective of strand z, going over strand y and then strand x is the same as going over strand x and then strand y.

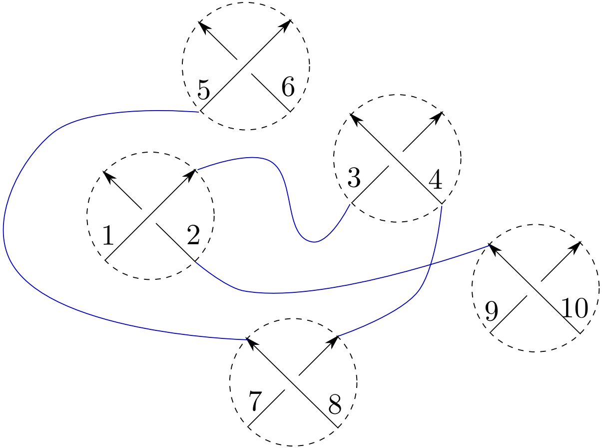
w-Tangles

Why should we care about w-tangles?

- Divide and conquer strategy.
- In the case of one component we obtain (long) w-knots, which includes (long) usual knots.
- The finite-type theory of w-tangles, a.k.a. *expansion*, is much simpler than for virtual tangles ([BND14, BND16]). The letter w stands for "welded", or "warm-up".

w-Tangles

We can generate any w-tangle by "stitching" a collection of crossings together.



Meta-monoids

Definition

A **meta-monoid** $\mathcal{G}$ (see [BNS13, BN15, Hal16]) is a collection of sets $(\mathcal{G}^X)$ indexed by finite sets X together with the following operations:

- "stitching" $m_z^{x,y} : \mathcal{G}^{\{x,y\}\cup X} \to \mathcal{G}^{\{z\}\cup X}$, where $\{x,y,z\} \cap X = \emptyset$ and $x \neq y$,
- "identity" $e_x : \mathcal{G}^X \to \mathcal{G}^{\{x\}\cup X}$, where $x \notin X$,
- "deletion" $\eta_x : \mathcal{G}^{X\cup\{x\}} \to \mathcal{G}^X$, where $x \notin X$,
- "disjoint union" $\sqcup : \mathcal{G}^X \times \mathcal{G}^Y \to \mathcal{G}^{X\cup Y}$, where $X \cap Y = \emptyset$,
- "renaming" $\sigma_z^x : \mathcal{G}^{X\cup\{x\}} \to \mathcal{G}^{X\cup\{z\}}$, where $\{x,z\} \cap X = \emptyset$.

satisfying the following axioms

- $m_u^{x,y} /\!\!/ m_v^{u,z} = m_u^{y,z} /\!\!/ m_v^{x,u}$ (*meta-associativity*),
- $e_a /\!\!/ m_c^{a,b} = \sigma_c^b$ (*left identity*),
- $e_b /\!\!/ m_c^{a,b} = \sigma_c^a$ (*right identity*),

Meta-monoids

Example. Given a monoid G with identity e (or an algebra), one obtains a meta-monoid as follows. Let

$$G^X = \{\text{functions } f : X \to G\}.$$

We write an element of G^X as $(x \to g_x, \dots)$, where $x \in X$ and $g_x \in G$. In the following operations, $\dots$ denotes the remaining entries, which stay unchanged:

$$\begin{aligned}
(x \to g_x, y \to g_y, \dots) \mathbin{/\!\!/} m_z^{x,y} &= (z \to g_x g_y, \dots), \\
(y \to g_y, \dots) \mathbin{/\!\!/} e_x &= (x \to e, y \to g_y, \dots), \\
(x \to g_x, y \to g_y, \dots) \mathbin{/\!\!/} \eta_x &= (y \to g_y, \dots), \\
(x \to g_x, \dots) \sqcup (y \to g_y, \dots) &= (x \to g_x, \dots, y \to g_y, \dots), \\
(x \to g_x, y \to g_y, \dots) \mathbin{/\!\!/} \sigma_z^x &= (z \to g_x, y \to g_y, \dots).
\end{aligned}$$

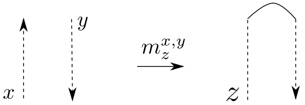
The axioms are straightforward to verify. ♣

Meta-monoids

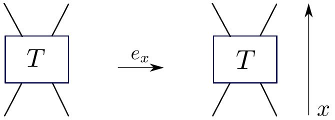
Our main object of study is the **meta-monoid $\mathcal{W}$ of w-tangles** defined as follows. For a finite set X, let

$$\mathcal{W}^X = \{\text{w-tangles with } |X| \text{ labeled components}\}.$$

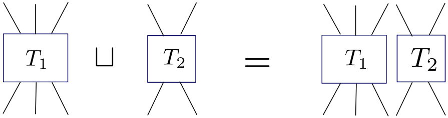
The operations are given by



stitching



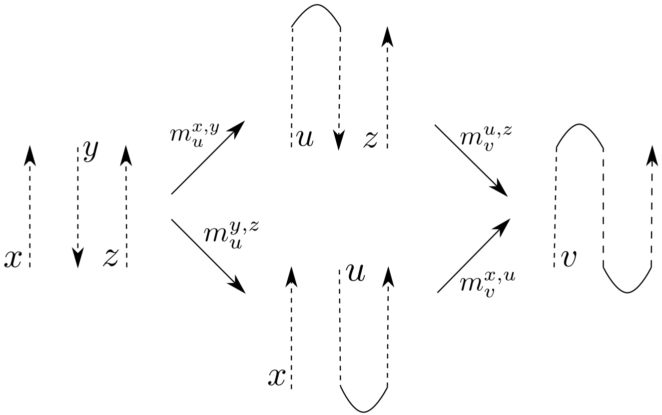
identity



disjoint union

Meta-monoids

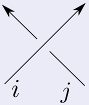
Let us verify meta-associativity

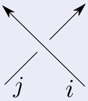


Meta-monoids

Proposition

The meta-monoid $\mathcal{W}$ is generated by the crossings

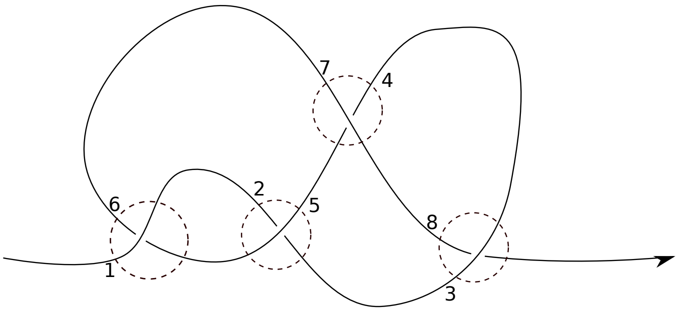
$R^{+}_{i,j}$ = 

; $R^{-}_{i,j}$ = 

i.e. all expressions that can be formed using the crossings and the meta-monoid operations, modulo the relations $R2$, $R3$, and OC.

Meta-monoids

For a concrete example, let us look at the long figure-eight knot:



With the labeling as in the above figure, the long figure-eight knot is given by

$$R^{+}_{1,6} R^{+}_{5,2} R^{-}_{3,8} R^{-}_{7,4} \mathbin{/\!\!/} m_1^{1,2} \mathbin{/\!\!/} m_1^{1,3} \mathbin{/\!\!/} m_1^{1,4} \mathbin{/\!\!/} m_1^{1,5} \mathbin{/\!\!/} m_1^{1,6} \mathbin{/\!\!/} m_1^{1,7} \mathbin{/\!\!/} m_1^{1,8}.$$

Γ-calculus

- Let us describe the meta-monoid Γ that will serve as the target space of an algebraic invariant for w-tangles.
- For a finite set of labels $X = \{a, b\} \cup S$, where $S \cap \{a, b\} = \emptyset$, let Γ^X be the collection of elements of the form

$$\left(\begin{array}{c|ccc} \omega & x_a & x_b & x_S \\ \hline y_a & \alpha & \beta & \theta \\ y_b & \gamma & \delta & \epsilon \\ y_S & \phi & \psi & \Xi \end{array}\right) = \left(\begin{array}{c|c} \omega & x_X \\ \hline y_X & M \end{array}\right).$$

- Here ω and each entry of M is a rational function in t_i's, $i \in X$. We call ω the **scalar part** and M the **matrix part**. We also require that $M|_{t_i \to 1} = I$, the identity matrix.
- Note that M has labeled rows and columns. So we have to specify the labels unless it is explicit from the context.

Γ-calculus

The two main operations of Γ-calculus are:
disjoint union

$$\left(\begin{array}{c|c} \omega_1 & x_{X_1} \\ \hline y_{X_1} & M_1 \end{array}\right) \sqcup \left(\begin{array}{c|c} \omega_2 & x_{X_2} \\ \hline y_{X_2} & M_2 \end{array}\right) = \left(\begin{array}{c|cc} \omega_1\omega_2 & x_{X_1} & x_{X_2} \\ \hline y_{X_1} & M_1 & \mathbf{0} \\ y_{X_2} & \mathbf{0} & M_2 \end{array}\right),$$

stitching

$$\left(\begin{array}{c|ccc} \omega & x_a & x_b & x_S \\ \hline y_a & \alpha & \beta & \theta \\ y_b & \gamma & \delta & \epsilon \\ y_S & \phi & \psi & \Xi \end{array}\right) /\!\!/ m_c^{a,b} = \left(\begin{array}{c|cc} (1-\gamma)\omega & x_c & x_S \\ \hline y_c & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ y_S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array}\right)_{t_a, t_b \to t_c}$$

Γ-calculus

Proposition

There is a **meta-monoid homomorphism** φ from the meta-monoid $\mathcal{W}$ to Γ-calculus, i.e. a collection of maps $(f : \mathcal{W}^X \to \Gamma^X)$ that commutes with the operations, given by

$$\varphi(R^{\pm}_{a,b}) = \left(\begin{array}{c|cc} 1 & x_a & x_b \\ \hline y_a & 1 & 1 - t_a^{\pm 1} \\ y_b & 0 & t_a^{\pm 1} \end{array}\right).$$

Γ-calculus

Proposition

Let K be a long knot and

$$\varphi(K) = \left(\begin{array}{c|c} \omega & x_1 \\ \hline y_1 & 1 \end{array} \right).$$

Then $\omega \doteq \Delta_{\widetilde{K}}(t)$. Here $\Delta_{\widetilde{K}}(t)$ is the Alexander polynomial of $\widetilde{K}$, where $\widetilde{K}$ is the closed knot obtained by closing the open component of K trivially and $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$.

So we see that Γ-calculus is an extension of the Alexander polynomial to tangles.

Γ-calculus

Moreover, for general w-tangle we have the following result.

Proposition

Let T be a w-tangle with scalar part ω and matrix part M, then ω is a Laurent polynomial and ωM is a matrix whose entries are Laurent polynomials.

Categorification?

Γ-calculus

So why should we care about Γ-calculus?

- It is polynomial time, with a simple implementation on a computer.
- It is an invariant of tangles, and we have explicit formulas for strand deletion, orientation reversal, and strand doubling.
- It contains the Gassner-Burau representation of braids/string links.
- It provides a framework to study the Alexander polynomial. Namely, to prove certain property, one just needs to check the crossings and show that the property is preserved under stitching.

Γ-calculus

Aside: Where does Γ-calculus come from?

Proposition

We have the following commutative diagram

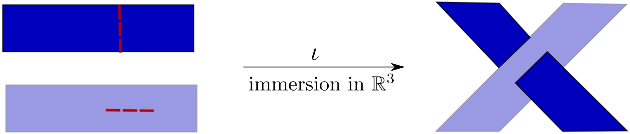
$$\begin{tikzcd}
\mathcal{W} \arrow[r, "Z"] \arrow[drr, "\psi"] \arrow[ddrr, "\varphi"'] & \mathcal{A}^w \arrow[r, "T_{\mathfrak{g}_0}"] & \mathbb{U}(\mathfrak{g}_0) \\
 & & \mathbb{G}_0 \arrow[u] \arrow[d] \\
 & & \Gamma
\end{tikzcd}$$

A part of this diagram has been generalized to provide a more powerful invariant (see [BN17]).

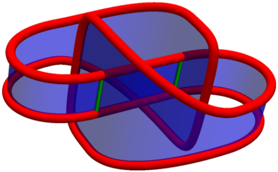
Ribbon Knots

Definition

A knot is called **ribbon** if it can be written as the boundary of a 2-disk that is immersed into S^3 with **ribbon singularities**.



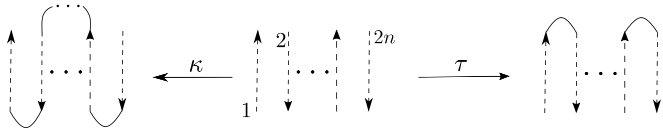
A ribbon singularity



A ribbon knot

Ribbon Knots

We have the following characterization of ribbon knots in terms of tangles. First we introduce the two **closure operations**.

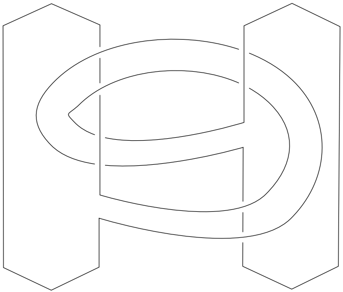


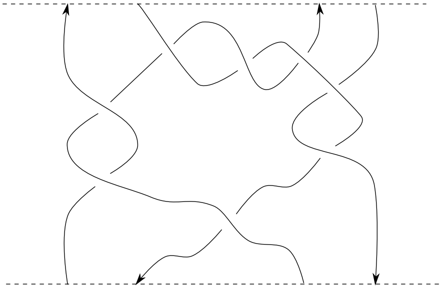
Proposition

A long knot K is ribbon if and only if there exists a $2n$-component pure up-down tangle T such that $\kappa(T)$ is the knot K and $\tau(T)$ is the trivial n-component tangle, i.e. it bounds n disjoint embedded half-disks in $\mathbb{R}^2$ as in the following figure.

Ribbon Knots

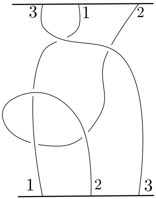
For example we have the following ribbon knot and its tangle presentation





Ribbon Knots

- Now we would like to present a "unitary result" for string links.
- Let T be a string link and suppose that for simplicity the bottom endpoints of T are labeled by $(1, 2, \ldots, n)$ and the top endpoints of T are labeled by $(\rho_1, \ldots, \rho_n)$



- Suppose

$$\varphi(T) = \left(\begin{array}{c|ccc} \omega & x_1 & \cdots & x_n \\ \hline y_1 & & & \\ \vdots & & M & \\ y_n & & & \end{array}\right).$$

We let M^ρ be the matrix whose ith column is the ρ_ith column of M.

Ribbon Knots

- We also let

$$\Omega = \begin{pmatrix} (1-t_1)^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_n)^{-1} \end{pmatrix}$$

- And let

$$\Omega(\rho) = \begin{pmatrix} (1-t_{\rho_1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{\rho_2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_{\rho_n})^{-1} \end{pmatrix}.$$

Ribbon Knots

Proposition [Vo17]

We have

$$(M^\rho)^* \Omega M^\rho = \Omega(\rho),$$

and

$$\overline{\omega} \doteq \omega \det(M^\rho).$$

As a corollary we obtain the classical Fox-Milnor condition for ribbon knots.

Proposition (Fox-Milnor Condition)

If a knot K is ribbon, then the Alexander polynomial of K, $\Delta_K(t)$ satisfies

$$\Delta_K(t) \doteq f(t) f(t^{-1}),$$

where $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$ and f is a Laurent polynomial.

Ribbon Knots

Some advantages of our proof of the Fox-Milnor condition

- Our proof uses only elementary linear algebra.
- As part of the proof the function f appears as an invariant of a tangle presentation of the ribbon knot.
- The original proof of the Fox-Milnor condition uses homology, which does not distinguish between ribbon knots and slice knots, whereas our proof proceeds directly from the characterization of ribbon knots in terms of tangles. Therefore we hope that a stronger version of Γ-calculus might shed some light on the slice-ribbon conjecture.

THANK YOU

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[Hal16] Iva Halacheva. *Alexander Type Invariants of Tangles*. `https://arxiv.org/abs/1611.09280`, 2016.

[Vo17] Huan Vo. *On Meta-Monoids and the Fox-Milnor Condition*. `https://arxiv.org/abs/1710.08993`, 2017.