On Γ-Calculus and the Fox-Milnor Condition

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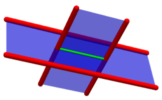
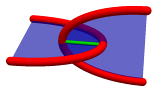
Abstract. In this talk we present a new proof of the Fox-Milnor condition, which is a statement about the Alexander polynomial of ribbon knots. Our approach is based on an extension of the Alexander polynomial to tangles, known as Γ-calculus. Since our proof utilizes a characterization of ribbon knots which does not apply to slice knots we hope that it has some potential for generalization in order to tackle the slice-ribbon conjecture.

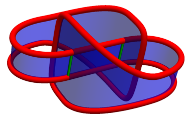
Ribbon knots. A knot is **ribbon** if it is the boundary of a smoothly immersed D^2 into S^3 with only **ribbon singularities**.



A ribbon singularity



A clasp singularity



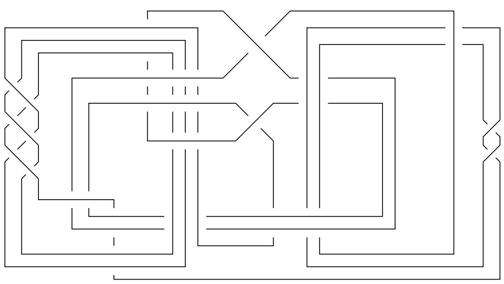
A ribbon knot

(pictures from [BN17])

Slice knots. A knot is **(smoothly) slice** if it is the boundary of a smoothly embedded D^2 into D^4. It is clear that a ribbon knot is slice.

Slice-ribbon Conjecture. Every slice knot is ribbon.

A potential counter example [GST10]:



The Fox-Milnor condition [FM66, Vo18]. If a knot K is ribbon, then the Alexander polynomial of K, $\Delta_K(t)$ satisfies

$$\Delta_K(t) \doteq f(t)f(t^{-1}),$$

where $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$ and f is a Laurent polynomial.

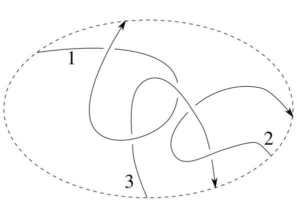
Some advantages of our approach.

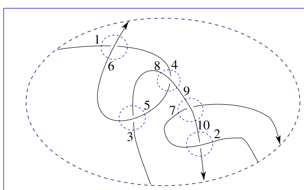
First. The original proof also works for slice knots. In contrast, our proof only works for ribbon knots, and so might have some potential for generalization to tackle the slice-ribbon conjecture.

Second. The bulk of our proof uses mainly elementary linear algebra.

Third. We have an interpretation of the function f as an invariant of a tangle presentation of the ribbon knot.

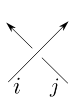
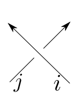
Tangles. A tangle is a piece of a knot. For instance



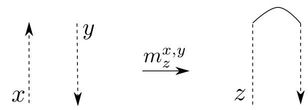


$R^+_{2,10}R^+_{9,7}R^-_{8,4}R^-_{5,3}R^-_{6,1} \,/\!/\, m_1^{1,4} \,/\!/\, m_1^{1,5} \,/\!/\, m_1^{1,6}$
$/\!/\, m_2^{2,7} \,/\!/\, m_3^{3,8} \,/\!/\, m_3^{3,9} \,/\!/\, m_3^{3,10}$

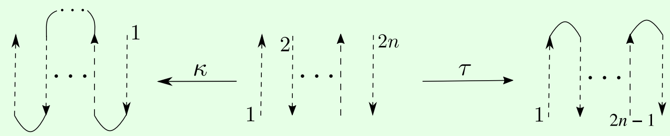
One can obtain a tangle using the **positive crossings** and **negative crossings**

$R^+_{i,j} =$  ; $R^-_{i,j} =$ 

together with the **stitching operation**

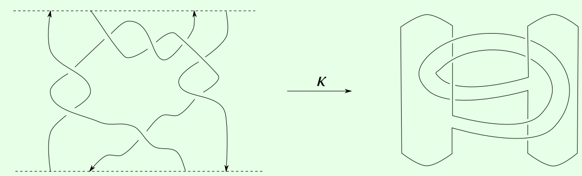


A tangle characterization of ribbon knots. Given a $2n$-component tangle, there are two closure operations: the **knot closure** κ and **tangle closure** τ



Theorem ([BN17, Hab06, Khe17, Vo18]). *A knot K is ribbon if and only if there exists a $2n$-component tangle $\mathcal{T}$ such that $\tau(\mathcal{T})$ is trivial and $\kappa(\mathcal{T}) = K$.*

Example.



Γ-calculus [BNS13]. We introduce a target space of an algebraic invariant for tangles known as **Γ-calculus**. For a tangle whose components are labeled by the elements of some set X its invariant has the form $\left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right)$, where ω, the **scalar part**, is a rational function in $\{t_i : i \in X\}$, and M, the **matrix part**, is a matrix whose entries are rational functions in $\{t_i : i \in X\}$. Note that the rows and columns of M are labeled by the elements of X. To define the invariant, we just need to specify the images of the crossings

$$R^{\pm}_{i,j} \mapsto \left(\begin{array}{c|cc} 1 & i & j \\ \hline i & 1 & 1-t_i^{\pm 1} \\ j & 0 & t_i^{\pm 1} \end{array}\right)$$

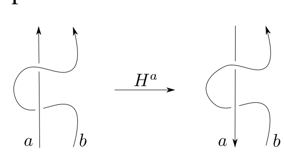
and the stitching operation

$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right) \xrightarrow[t_a,t_b\to t_c]{m_c^{a,b}} \left(\begin{array}{c|cc} (1-\gamma)\omega & c & S \\ \hline c & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array}\right).$$

Some properties of Γ-calculus. The Γ-calculus invariant of tangles satisfies the following properties.

- When we set all t_i to 1, then $\omega = 1$ and $M = I$.
- The sum of the entries in each column of M is 1.
- ω is a Laurent polynomial and ωM is a matrix whose entries are Laurent polynomial.
- For a one-component tangle, i.e. a long knot K, the matrix part is 1 and the scalar part is the Alexander polynomial $\Delta_K(t)$ (up to multiplication by $\pm t^{\pm n}$, $n \in \mathbb{Z}$). In particular Γ-calculus is an extension of the Alexander polynomial to tangles.

Orientation reversal [BN14]. It is useful to have a formula for the **orientation reversal** operation

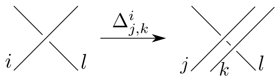


In Γ-calculus it is given by

$$\left(\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array}\right) \xrightarrow{dH^a} \left(\begin{array}{c|cc} \alpha\omega/s_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha \end{array}\right).$$

For the definition of s_a, see [Vo18].

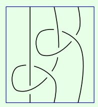
Strand doubling [BN14]. Let us also consider the **strand doubling** operation.



In Γ-calculus it is given by

$$\left(\begin{array}{c|cc} \omega & i & S \\ \hline i & \alpha & \theta \\ S & \phi & \Xi \end{array}\right) \to \left(\begin{array}{c|ccc} \omega & j & k & S \\ \hline j & \frac{-\alpha+t_jt_ks_i+t_j\nu}{\mu} & \frac{(-1+t_j)\nu}{\mu} & \frac{(-1+t_j)\theta}{\mu} \\ k & \frac{t_j(-1+t_k)\nu}{\mu} & \frac{-s_i+t_jt_k\alpha-t_j\nu}{\mu} & \frac{t_j(-1+t_k)\theta}{\mu} \\ S & \phi & \phi & \Xi \end{array}\right)$$

where $\mu = -1+t_i$ and $\nu = \alpha - s_i$.

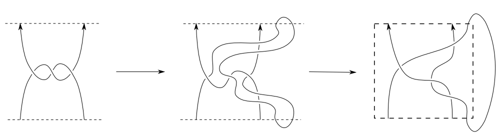
Unitary property. Let $\mathcal{T}$ be a **string link**  and suppose its image in Γ-calculus is $\left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right)$.

Theorem ([KLW01, Vo18]). The matrix part M satisfies $M^*\Omega M = \Omega$, where $M^* = (M^T)_{t_i\to t_i^{-1}}$ and

$$\Omega = \begin{pmatrix} (1-t_1)^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_2)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_n)^{-1} \end{pmatrix},$$

and the scalar part ω satisfies $\overline{\omega} \doteq \omega \det(M)$, where $\overline{\omega}$ denotes the conjugation $t_i \to t_i^{-1}$.

Proof of unitary property (sketch). We first observe that a string link can be obtained from a braid by a sequence of *appropriate* stitching operations. This follows from the fact that we can eliminate a downward arc using the following procedure:



Therefore our strategy is to verify the unitary property for braids and then show that it still holds under *appropriate* stitching operations. We first show the unitary property for the matrix part and then use it to show the unitary property for the scalar part. The key point is we can decompose the stitching operation into a sequence of operations as follows

$$\begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} \to \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma-1 \end{pmatrix} \to \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \frac{\epsilon}{\gamma-1} & \frac{\delta}{\gamma-1} & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} & \mathbf{0} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} & 0 \\ \frac{\epsilon}{\gamma-1} & \frac{\delta}{\gamma-1} & 1 \end{pmatrix}.$$

Note that except for the first one, all the operations are simply elementary row operations.

Proof of the Fox-Milnor condition (sketch). Our strategy is to simply express the characterization of ribbon knots in the language of Γ-calculus. So let $\mathcal{T}$ be a tangle presentation of the ribbon K and suppose that the image of $\mathcal{T}$ in Γ-calculus is

$$\left(\begin{array}{c|cc} \omega & \textbf{\textit{odd}} & \textbf{\textit{even}} \\ \hline \textbf{\textit{even}} & \gamma & \delta \\ \textbf{\textit{odd}} & \alpha & \beta \end{array}\right).$$

The image of $\mathcal{T}$ under the τ closure is

$$\left(\begin{array}{c|cc} \omega & \textbf{\textit{odd}} & \textbf{\textit{even}} \\ \hline \textbf{\textit{even}} & \gamma & \delta \\ \textbf{\textit{odd}} & \alpha & \beta \end{array}\right) \xrightarrow[t_{even}\to t_{odd}]{\tau = m^{odd,even}_{odd}} \left(\begin{array}{c|c} \omega\det(I-\gamma) & \textbf{\textit{odd}} \\ \hline \textbf{\textit{odd}} & \beta+\alpha(I-\gamma)^{-1}\delta \end{array}\right).$$

Then from the characterization of ribbon knots we have

$$\omega\det(I-\gamma) = 1 \text{ and } \beta + \alpha(I-\gamma)^{-1}\delta = I.$$

Now under the κ closure $\mathcal{T}$ becomes a long knot and its scalar part is

$$\omega \det\left(\alpha + (I-\beta)\delta^{-1}\left(\gamma - \begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}\right)\right)\det(\delta).$$

Using $\alpha(I-\gamma)^{-1}\delta = I-\beta$ and $\omega = 1/\det(I-\gamma)$ we can rewrite the above as

$$\omega\det(\alpha)\omega\det(\delta).$$

From assumption since the κ closure of $\mathcal{T}$ is the knot K we have

$$\Delta_K(t) \doteq \omega\det(\alpha)\omega\det(\delta)|_{t_i\to t}.$$

To proceed we reverse the orientations of the even strands to obtain

$$\left(\begin{array}{c|cc} \omega & \textbf{\textit{even}} & \textbf{\textit{odd}} \\ \hline \textbf{\textit{even}} & \delta & \gamma \\ \textbf{\textit{odd}} & \beta & \alpha \end{array}\right) \xrightarrow[t_{even}\to t_{even}^{-1}]{dH^{even}} \left(\begin{array}{c|cc} \omega\det(\delta) & \textbf{\textit{even}} & \textbf{\textit{odd}} \\ \hline \textbf{\textit{even}} & \delta^{-1} & \delta^{-1}\gamma \\ \textbf{\textit{odd}} & -\beta\delta^{-1} & \alpha-\beta\delta^{-1}\gamma \end{array}\right).$$

Then the unitary property of the scalar part tells us that

$$\omega\det(\delta)\Big|_{t_{odd}\to t_{odd}^{-1}} \doteq \omega\det(\delta)\det\begin{pmatrix} \alpha-\beta\delta^{-1}\gamma & -\beta\delta^{-1} \\ \delta^{-1}\gamma & \delta^{-1} \end{pmatrix}\Bigg|_{t_{even}\to t_{even}^{-1}}.$$

Taking $t_{even} \to t_{even}^{-1}$ in both sides we obtain

$$\begin{aligned} \overline{\omega\det(\delta)} &\doteq \omega\det(\delta)\det\begin{pmatrix} \alpha-\beta\delta^{-1}\gamma & -\beta\delta^{-1} \\ \delta^{-1}\gamma & \delta^{-1} \end{pmatrix} \\ &= \omega\det(\delta)\det(\alpha-\beta\delta^{-1}\gamma+\beta\delta^{-1}\delta\delta^{-1}\gamma)\det(\delta^{-1}) \\ &= \omega\det(\alpha). \end{aligned}$$

Finally letting $f(t) = \omega\det(\delta)|_{t_i\to t}$, we obtain

$$\Delta_K(t) \doteq f(t)f(t^{-1}),$$

which is precisely the Fox-Milnor condition.

Master diagram. For interested readers, please refer to [Vo18] for more details. The formulas for Γ-calculus are obtained via a technology known as **expansion**, which is a map from the meta-monoid of w-tangles $\mathcal{W}$ to its associated graded $\mathcal{A}^w$, the meta-monoid of arrow diagrams. To work with arrow diagrams we choose a particular Lie algebra, namely $\mathfrak{g}_0$, and convert an arrow diagram to an element of the universal enveloping algebra of $\mathfrak{g}_0$. We then work with formulas involving exponentials in the universal enveloping algebra of $\mathfrak{g}_0$ using techniques from quantum mechanics in order to arrive at Γ. In brief we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{W} \arrow[r, "Z"] \arrow[drr, "\psi"] \arrow[ddrr, "\varphi"'] & \mathcal{A}^w \arrow[r, "T_{\mathfrak{g}_0}"] & \mathbb{U}(\mathfrak{g}_0) \\ & & \mathbb{G}_0 \arrow[u, "\iota"] \arrow[d, "\eta"] \\ & & \Gamma \end{tikzcd}$$

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Thank You.