BRAID CALCULATOR

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Abstract. This paper aims to give an overview about solutions to the word problem for the braid group B_n over the disc and over the 2-sphere. Using MATLAB, we created computer software called 'Braid Calculator'. Our Braid Calculator has a graphical user interface with several buttons for simple calculations in braid groups.

Keywords: braid group, braid word problem, Artin combing algorithm.

1 INTRODUCTION

Braid groups were introduced explicitly by Emil Artin in 1925. Since then, there have been many developments and generalizations in the field. Braid groups turn out to be useful in different areas of mathematics and engineering like algebraic geometry, cryptography or even robotics.

Braid groups can be given by generators and relations. Thus the word problem is of particular importance. The word problem is to decide whether two different words represent the same braid or, equivalently, whether a given braid word represents the identity braid. There are many ways to solve the word problem for braids over the disc. We implemented two of them in our Braid Calculator: Artin representation and Artin combing algorithm. Also, with some modifications to the Artin combing algorithm, we can solve the word problem for braids over the 2-sphere.

2 BACKGROUNDS

2.1 BRAIDS OVER THE DISC

There are a few ways to define braids. The most intuitively clear approach is to think of an n-braid as a collection of n strands in (x, y, t)-space. The strands are disjoint and monotone in the t direction. The endpoints of the strands are fixed. Two braids are considered to be equivalent if one of them can be continuously deformed to the other one, with endpoints fixed throughout the deformation.

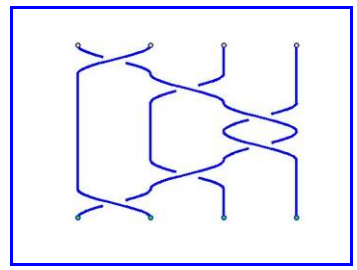

Figure 1: Example of a 4-braid.

Consider the set of all n-braids. The multiplication of two braids is given by concatenation. Specifically, a braid A is multiplied to a braid B by putting A on top of B. The set of all n-braids forms a group under this multiplication, which we denote by B_n. The identity braid (the trivial braid) is the braid where all the strands go straight along the t direction. The inverse of a braid is its reflection about the horizontal axis.

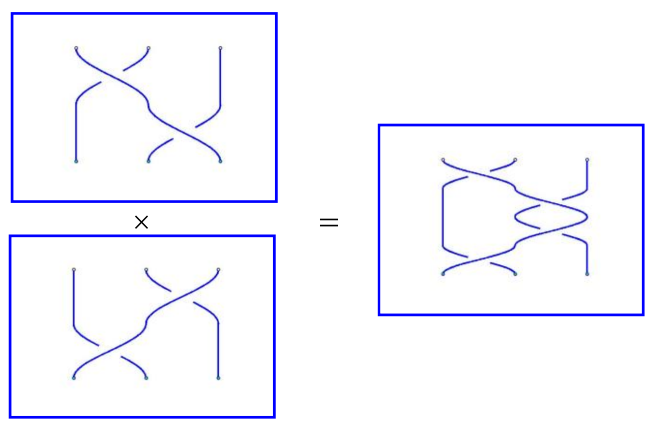

Figure 2: Multiplication of two braids.

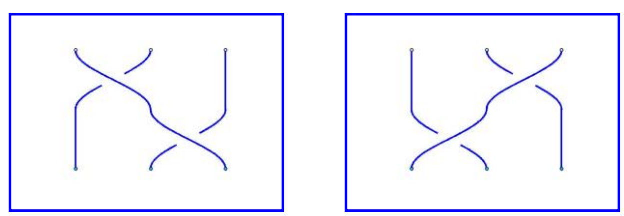

Figure 3: Braid A and its inverse.

2.2 ARTIN PRESENTATION OF B_n

Artin introduced a presentation of the braid group B_n in 1925. Each n-braid can be defined abstractly as a word in $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$. Each generator can be visualized as a braid where all the strands go straight, except for two adjacent strands which cross over each other.

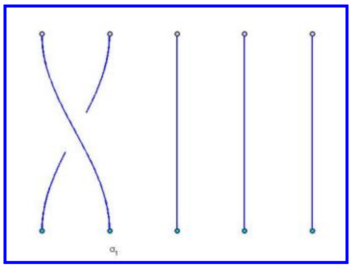

Figure 4: The generator σ_1 of 5-braids.

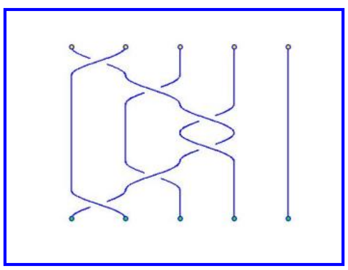

Figure 5: The 5-braid $\sigma_1^{-1}\sigma_2\sigma_3^2\sigma_2^{-1}\sigma_1$.

These generators are subject to the relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1,$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

2.3 BRAIDS AND PERMUTATIONS

Each n-braid defines a permutation of n points. For example, if we number the endpoints of a braid $1,2,...,n$, starting from the left, then the braid in figure 5 corresponds to the trivial permutation (the permutation where each point is sent to itself). The map that sends each n-braid to its corresponding permutation is a homomorphism, with the kernel P_n, known as the pure braid group. Each element of P_n is called a pure braid (a braid which has the trivial permutation). The pure braid group plays an important role in the solution to the word problem.

2.4 ARTIN REPRESENTATION OF B_n

Each n-braid can be thought of as an automorphism of the free group in n generators. Specifically, Artin representation is the map defined as follows. It sends a generator σ_i to the free group automorphism acting by

$$x_i \to x_{i+1}$$
$$x_{i+1} \to x_{i+1}^{-1}x_i x_{i+1}$$
$$x_j \to x_j,\ j \neq i, i+1.$$

2.5 BRAIDS OVER THE 2-SPHERE

In definitions above we considered braids over the disc, i.e., the endpoints of a braid are fixed on the disc and a braid is sandwiched between two parallel discs. For braids over the 2-sphere, the endpoints of a braid are fixed points on the 2-sphere instead and a braid is sandwiched between two concentric 2-spheres. Braids over the 2-sphere can be thought of as braids on the disc, with one additional relation:

$$\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 = 1.$$

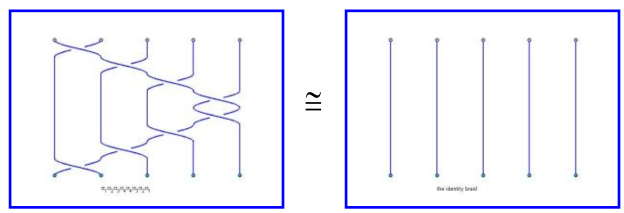

Figure 6: The relation $\sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1 = 1.$

2.6 THE BRAID WORD PROBLEM

Recall that the braid word problem is to decide whether two different words represent the same braid or, equivalently, whether the given braid word equals the identity. For example, the two words $\sigma_1\sigma_3$ and $\sigma_3\sigma_1$ represent the same braid, as seen in the following figure:

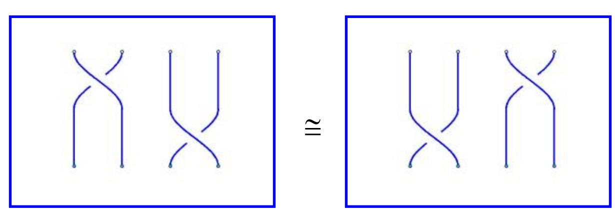

Figure 7: The braid represented by $\sigma_1\sigma_3$ and $\sigma_3\sigma_1$.

The word problem for braids over the disc was first solved by Artin in 1925. Several faster and more efficient algorithms were developed later. However, we use Artin's algorithm in this work because it allows us to solve the word problem for braids over the 2-sphere after some modifications.

3 METHODOLOGY

3.1 SOLUTION TO THE WORD PROBLEM FOR BRAIDS OVER THE DISC

Artin himself gave two ways to solve the word problem for braids over the disc. One way is to look at the Artin representation of a braid. A braid is trivial if and only if its Artin representation is the identity automorphism.

Another way is to express each n braid uniquely into a special form, called its Artin normal form. Thus two braids are equal if and only if they have the same Artin normal form. The process to obtain the Artin normal form of a braid is known as Artin combing. A detailed treatment of the procedure can be found in [2]. Here are the main steps to obtain the Artin normal form for braids over the disc.

First, let's notice that it suffices to consider only pure braids. Indeed, given two braids α and β, deciding if they are equal is the same as deciding if the braid $\alpha\beta^{-1}$ is trivial. But the trivial braid is pure, so if $\alpha\beta^{-1}$ is not pure, then, in particular, it is non-trivial either. The Artin normal form of a braid $b \in B_n$ is a product $b_1 b_2 \cdots b_{n-1}$, where each b_i is a element of the free group F_i embedded into the braid group in a certain way.

Also, the Artin normal form of a braid can be expressed as a word in a new set of generators $\{a_{i,j} : n \geq i > j \geq 1\}$. Each $a_{i,j}$ can be visualized as a braid with all strands straight, except that the i th strand goes above all other strands, up to the j th position, wraps around the j th strand and goes back above all other strands, to the i th position.

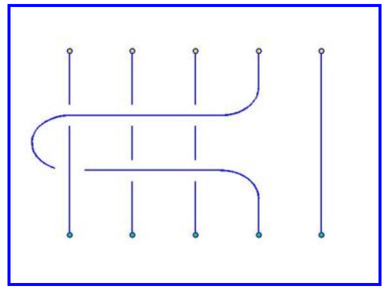

Figure 8: The Artin combing generator $a_{4,1}$.

There are some relations between $a_{i,j}$'s, but the Artin normal form is unique. We can convert a word in $a_{i,j}$'s into a word in σ_i's by

$$a_{i,j} = \sigma_i^{-1}\sigma_{i-1}^{-1} \cdots \sigma_j^{-2}\sigma_{j+1} \cdots \sigma_{i-1}\sigma_i.$$

For example, the Artin normal form of the 5-braid $\sigma_1\sigma_2^2\sigma_1$ is $a_{2,1}^{-1}a_{3,2}^{-1}a_{3,1}^{-1}a_{3,2}$

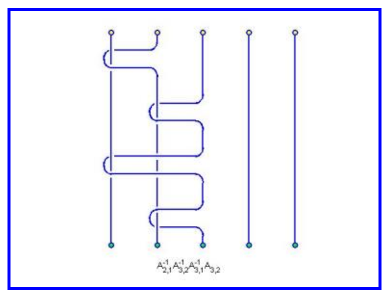

Figure 9: The Artin normal form of the 5-braid $\sigma_1\sigma_2^2\sigma_1$.

3.2 SOLUTION TO THE WORD PROBLEM FOR BRAIDS OVER THE 2-SPHERE

We apply similar approach to solve the word problem for braids over the 2-sphere. First, it is known that the world problem for 2-braids and 3-braids over the 2-sphere is trivial since both groups are finite. Thus we need to consider braids with at least 4 strands.

With the extra relation, the normal form for braids over the 2-sphere differs slightly from that for braids over the plane or disc. Here, the Dirac braid given by

$$\Delta_n = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$$

is of particular importance. Visually, Δ_n is the full or 2π-twist of the trivial braid. The normal form of a braid with at least 4 strands over the 2-sphere is

$$b = b_3 b_4 \cdots b_{n-1} \Delta_n^p,$$

where b_i is a word in F_{i-1} for $i \geq 4$, p is either 0 or 1, the first strand goes trivially apart from the rest (though it might be included into Δ_n), second and third strands go straight. The reason for p being either 0 or 1 is that $\Delta_n^2 = 1$.

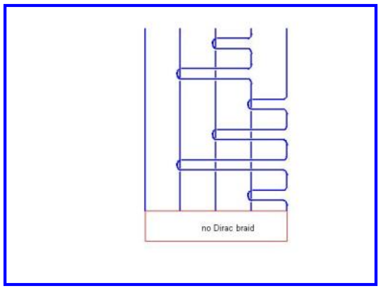

Figure 10: The normal form of the braid $\sigma_1\sigma_2^2\sigma_1$ over the 2-sphere.

4 BRAID CALCULATOR

Braid calculator is a GUI (graphical user interface) created using MATLAB to facilitate investigation of braids. One can input a braid either as a word in σ_i's or in $a_{i,j}$'s. The input is a vector and each generator is represented by its subscript, together with the minus sign to indicate the inverse. The GUI will draw the picture of the braid, compute the Artin representation, permutation of a braid, Artin normal form, normal form of a braid over the 2-sphere. It can also perform the face operator (removing an arbitrary strand) and degeneracy operator (doubling an arbitrary strand).

Figure 11: Braid calculator.

5 CONCLUSION

Although the Artin combing algorithm is not very efficient, it reveals some additional nice features of the braid group and also has a direct implication for the solution to the word problem for braids over the 2-sphere. We can use these normal forms to study Brunnian braids (a Brunnian braid is one that becomes trivial if any of its strands is removed). Brunnian braids occur in relation to homotopy groups of spheres, that is, an old and famous problem in algebraic topology. Specifically, there is the following exact sequence:

$$1 \to \mathrm{Brun}_{n+1}(S^2) \to \mathrm{Brun}_n(D^2)$$
$$\to \mathrm{Brun}_n(S^2) \to \pi_{n-1}(S^2) \to 1$$

discovered in [1].

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr Fedor Duzhin, for his constant support and patient guidance through every stage of the research process. Gratitude also goes to Asst Prof Andrew James Kricker for his helpful advices and willingness to answer and clarify my questions regarding the subject. Last but not least, I would like to thank my collaborator, Ng Wen Zheng Terence, who has worked with me through out the project and has provided many brilliant ideas.

We wish to acknowledge the funding support for this project from Nanyang Technological University under the Undergraduate Research Experience on Campus (URECA) programme.

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