# REPRESENTATIONS OF $\mathfrak{sl}(2,\mathbb{C})$ AND CLEBSCH-GORDAN RULE

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Abstract. In this notes we describe the Clebsch-Gordan rule for $\mathfrak{sl}(2,\mathbb{C})$, which gives a decomposition of tensor products of irreducible representations into irreducibles.

## 1. Irreducible Representations of $\mathfrak{sl}(2,\mathbb{C})$

Recall that the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is

$$\mathfrak{sl}(2,\mathbb{C}) = \{A \in \mathsf{Mat}_2(\mathbb{C}) \colon \operatorname{tr}(A) = 0\},$$

It has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Lie brackets are given by $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$.

By a ***representation*** $V$ of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$, the Lie algebra of all linear transformations of some complex vector space $V$. A subspace of $V$ that is stable under the action of $\mathfrak{g}$ is called a ***subrepresentation*** of $V$. A representation $V$ is ***irreducible*** if its only subrepresentations are either 0 or $V$ and is called ***completely reducible*** if it is the direct sum of irreducible representations. We would like to classify representations of $\mathfrak{sl}(2,\mathbb{C})$. For that, we first classify the irreducible representations. Some terminologies are in order.

Let $V$ be a representation of $\mathfrak{sl}(2,\mathbb{C})$. For $\lambda \in \mathbb{C}$, let

$$V[\lambda] \colon = \{v \in V \colon hv = \lambda v\}.$$

If $V[\lambda] \neq 0$, then $\lambda$ is called a ***weight*** of $V$, $V[\lambda]$ is called a ***weight space*** and elements of $V[\lambda]$ are called ***weight vectors***. A weight $\lambda$ of $V$ is called a ***highest weight*** of $V$ if

$$\operatorname{Re}\lambda \geq \operatorname{Re}\lambda' \quad \text{for any weight } \lambda' \text{ of } V.$$

If $\lambda$ is a highest weight of $V$, we call elements of $V[\lambda]$ ***highest weight vectors***. Note that any finite dimensional representation has a highest weight since $h$ always has at least one eigenvalue (over $\mathbb{C}$) and only has finitely many of them.

**Lemma 1.1.** *The actions of $e$ and $f$ on $V[\lambda]$ is given by*

$$eV[\lambda] \subset V[\lambda+2],$$
$$fV[\lambda] \subset V[\lambda-2].$$

*Proof.* The proof is a straight-forward computation based on the fact that $V$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$. Pick any $v \in V[\lambda]$, then

$$h(ev) = ([h,e] + eh)v = 2ev + \lambda ev = (\lambda+2)ev,$$
$$h(fv) = ([h,f] + fh)v = -2fv + \lambda fv = (\lambda-2)fv,$$

as required. $\square$

**Lemma 1.2.** *Let $V$ be a representation of $\mathfrak{sl}(2,\mathbb{C})$ with highest weight $\lambda$ and $v_0 \in V[\lambda]$ a highest weight vector. Define*

$$v_k = f^k v_0, \quad k \geq 0.$$

*Then*

(i) $ev_k = k(\lambda - k + 1)v_{k-1}$ *for* $k > 0$ *and* $ev_0 = 0$.
(ii) $hv_k = (\lambda - 2k)v_k$,

*Proof.* We prove part (i) and (ii) simultaneously by induction. When $k = 0$, by definition $hv_0 = \lambda v_0$. If $ev_0 \neq 0$, then by Lemma 1.1 we have $h(ev_0) = (\lambda + 2)ev_0$, contradicting the assumption that $\lambda$ is a highest weight. Thus $ev_0 = 0$. When $k = 1$,

$$hv_1 = hfv_0 = (\lambda - 2)fv_0 = (\lambda - 2)v_1$$

again by Lemma 1.1 and

$$ev_1 = efv_0 = ([e,f] + fe)v_0 = hv_0 = \lambda v_0,$$

where we use $ev_0 = 0$. Now suppose that (i) and (ii) are true for $k \geq 1$. We compute

$$hv_{k+1} = hfv_k = (\lambda - 2k - 2)fv_k = (\lambda - 2k - 2)v_{k+1},$$

where in the second equality we use the fact that $v_k$ is an eigenvector of $h$ with eigenvalue $\lambda - 2k$. Finally,

$$\begin{aligned} ev_{k+1} &= efv_k = ([e,f] + fe)v_k = hv_k + fev_k \\ &= (\lambda - 2k)v_k + k(\lambda - k + 1)fv_{k-1} = (k+1)(\lambda - k)v_k, \end{aligned}$$

as required. □

Now we are ready for our main theorem. In brief, there is a 1-1 correspondence between irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ and non-negative integers.

**Theorem 1.3.** *(1) For any $n \geq 0$, let $V_n$ be the finite dimensional vector space with basis $\{v_0, v_1, \ldots, v_n\}$. Define the action of $\mathfrak{sl}(2,\mathbb{C})$ by*

$$hv_k = (n - 2k)v_k, \quad 0 \leq k \leq n$$

$$fv_k = v_{k+1}, \quad 0 \leq k < n; \quad fv_n = 0,$$

$$ev_k = k(n + 1 - k)v_{k-1}, \quad 0 < k \leq n; \quad ev_0 = 0.$$

*Then $V_n$ is an irreducible representation of $\mathfrak{sl}(2)$, we will call it the* **irreducible representation with highest weight** *$n$.*

*(2) For $n \neq m$, representations $V_n$, $V_m$ are not isomorphic.*

*(3) Every finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to one of the representations $V_n$.*

*Proof.* (1) Notice that if $n = 0$, then $V_0 = \operatorname{span}\{v_0\}$ is the trivial representation. So assume that $n > 0$. To check that $V_n$ is indeed a representation of $\mathfrak{sl}(2,\mathbb{C})$, it suffices to check that

$$hv_k = (ef - fe)v_k; \quad 2ev_k = (he - eh)v_k; \quad -2fv_k = (hf - fh)v_k$$

for $k = 0, 1, \dots, n$. We have

$$\begin{aligned}(ef - fe)v_k &= ev_{k+1} - k(n-k+1)fv_{k-1} \\ &= (k+1)(n-k)v_k - k(n-k+1)v_k \\ &= (n-2k)v_k = hv_k.\end{aligned}$$

Note that the above computation also holds when $k = 0$ or $k = n$. Similarly,

$$\begin{aligned}(he - eh)v_k &= k(n-k+1)hv_{k-1} - (n-2k)ev_k \\ &= k(n-k+1)(n-2k+2)v_{k-1} - (n-2k)k(n-k+1)v_{k-1} \\ &= 2k(n-k+1)v_{k-1} = 2ev_k.\end{aligned}$$

The above computation also works when $k = 0$. Finally,

$$\begin{aligned}(hf - fh)v_k &= hv_{k+1} - (n-2k)fv_k \\ &= (n-2k-2)v_{k+1} - (n-2k)v_{k+1} \\ &= -2v_{k+1} = -2fv_k.\end{aligned}$$

Again the above computation also holds when $k = n$. Thus $V_n$ is a representation of $\mathfrak{sl}(2, \mathbb{C})$. Now to see that $V_n$ is irreducible, let $0 \neq W \subset V_n$ be a subrepresentation of $V$. Pick $0 \neq w \in W$ and suppose that

$$w = a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n, \quad a_m \neq 0,\ m \geq 0.$$

Then

$$e^n f^{n-m} w = \alpha a_m v_0,$$

where $\alpha a_m \neq 0$, as can easily be checked. Thus $v_0 \in W$, and because $W$ is a representation, it follows that $W = V_n$, as required.

(2) For $n \neq m$, representations $V_n$ and $V_m$ have different dimensions. Therefore they are not isomorphic.

(3) Now suppose that $V$ is a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. Then $V$ has a highest weight $\lambda$ with a highest weight vector $v_0$. Put

$$v_k = f^k v_0, \quad k \geq 0.$$

Then the actions of $\mathfrak{sl}(2, \mathbb{C})$ on $\{v_k\}$ are as in Lemma 1.2. If the $v_k$'s are non-zero, then they are linear independent, since they have different weights. Since $V$ is finite dimensional, there must exist some $n \geq 0$ such that $v_n \neq 0$ and $fv_n = 0$. So in particular $v_k = 0$ for all $k > n$. We have from Lemma 1.2

$$0 = ev_{n+1} = (n+1)(\lambda - n)v_n.$$

Since $v_n$ is non-zero, it follows that $\lambda = n$. So $V$ has highest weight $n$. The vectors $v_0, \dots, v_n$ are linearly independent. Now consider

$$W = \operatorname{span}\{v_0, \dots, v_n\}.$$

The action of $\mathfrak{sl}(2, \mathbb{C})$ on $W$ is exactly as given in part (1) of the theorem. So in particular $W$ is a subrepresentation of $V$ which is isomorphic to $V_n$. Since $V$ is irreducible, it follows that $V = W \cong V_n$, as required. $\square$

**Corollary 1.4.** *If $V_n$ is the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with highest weight $n$, then $V_n$ is the direct sum of its weight spaces:*

$$V_n = V[-n] \oplus V[-n+2] \oplus \cdots \oplus V[n-2] \oplus V[n],$$

*and each weight space is one dimensional. More specifically, $V[n-2k]$ is generated by $v_k$, where $v_k = f^k v_0$ for some highest weight vector $v_0$.*

## 2. Concrete Expressions for $V_n$

In this section we give a concrete expression for the irreducible representations $V_n$. Observe that $V_1 \cong \mathbb{C}^2$, the standard representation of $\mathfrak{sl}(2,\mathbb{C})$. The isomorphism is

$$v_0 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: z_1, \quad v_1 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} =: z_2.$$

Consider $S^n V_1$, the $n$th symmetric power of $V_1$, i.e. an element $P \in S^n V_1$ is a homogeneous polynomial in $z_1$ and $z_2$ of total degree $n$:

$$P = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \cdots + a_{n-1} z_1 z_2^{n-1} + a_n z_2^n,$$

where $a_0, \ldots, a_n \in \mathbb{C}$.

**Theorem 2.1.** *With the following actions:*

$$(h \cdot P)(z_1, z_2) = z_1 \frac{\partial P}{\partial z_1} - z_2 \frac{\partial P}{\partial z_2},$$
$$(e \cdot P)(z_1, z_2) = z_1 \frac{\partial P}{\partial z_2},$$
$$(f \cdot P)(z_1, z_2) = z_2 \frac{\partial P}{\partial z_1},$$

*we have $S^n\mathbb{C}^2 \cong V_n$ as representations of $\mathfrak{sl}(2,\mathbb{C})$. Thus $S^{\cdot}(V_1)$, the symmetric algebra of $V_1$, contains all irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$.*

*Proof.* Note that $S^n\mathbb{C}^2$ has the following basis

$$\left\{ \frac{1}{n!} z_1^n, \frac{1}{(n-1)!} z_1^{n-1} z_2, \ldots, z_1 z_2^{n-1}, z_2^n \right\}.$$

Let $v_k = \frac{1}{(n-k)!} z_1^{n-k} z_2^k$, $0 \le k \le n$. To see that $S^n\mathbb{C}^2 \cong V_n$, we just need to check that the actions of $e$, $f$, $h$ on the $v_k$'s follow the rule of part (1) of Theorem 1.3, which is a straight-forward exercise. $\square$

## 3. Digression on Lie Groups

For our purpose, it suffices to look at matrix Lie groups, i.e. closed subgroups of $\mathsf{GL}(n,\mathbb{C})$. For instance, the Lie group $\mathsf{SU}(2)$ is given by

$$\mathsf{SU}(2) = \left\{ A \in \mathsf{GL}(2,\mathbb{C}) \colon A^*A = I_2,\ \det(A) = 1 \right\}.$$

Here $A^*$ denotes the conjugate transpose of $A$. The group $\mathsf{SU}(2)$ is connected. It is homeomorphic to the three-dimensional sphere $\mathbb{S}^3$ sitting inside $\mathbb{R}^4$ and hence is compact and is simply connected.

Given a matrix Lie group $G$, its Lie algebra $\mathfrak{g}$ can be identified with the tangent space to $G$ at the identity $e$. One can obtain $\mathfrak{g}$ via the exponential mapping. More specifically, the Lie algebra $\mathfrak{g}$ is the set of all matrices $X$ such that $e^{tX} \in G$ for all real numbers $t$, where

$$e^{tX} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k X^k.$$

For instance, the Lie algebra of $\mathsf{SU}(2)$ is $\mathfrak{su}(2)$, given by

$$\mathfrak{su}(2) = \{A \in \mathsf{Mat}_2(\mathbb{C}) \colon A^* = -A,\ \operatorname{tr}(A) = 0\}.$$

Notice that $\mathfrak{su}(2)$ is not a complex vector space since if $A^* = -A$, then $(iA)^* = -iA^* = iA$. Its complexification is precisely the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, i.e.

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) + i\mathfrak{su}(2).$$

So in particular the complex representations of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$ are the same.

Given a representation of a matrix Lie group $G$, i.e. a smooth group homomorphism $G \to \mathsf{GL}(V)$ for some complex vector space $V$, one can obtain a representation of the corresponding Lie algebra $\mathfrak{g}$ as follows. For $X \in \mathfrak{g}$ and $v \in V$, we have

$$X \cdot v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e^{tX} \cdot v).$$

However, not every representation of $\mathfrak{g}$ comes from a representation of $G$. Nevertheless, it turns out that if $G$ is connected and simply-connected, then the category of representations of $G$ and the category of representations of $\mathfrak{g}$ are equivalent. So in our case in particular, the representation theory of $\mathsf{SU}(2)$ is the same as the representation theory of $\mathfrak{sl}(2, \mathbb{C})$.

Here we would like to introduce a notion that will be useful in the next section. Let $V$ be a representation of $G$ and suppose that $V$ is equipped with an ***invariant inner product*** $\langle \cdot, \cdot \rangle$, i.e.

$$\langle gv, gw \rangle = \langle v, w \rangle \quad \text{for all } g \in G \text{ and } v, w \in V. \tag{1}$$

In other words, $g$ acts as a unitary operator. One advantage of having an invariant inner product on $V$ is that if $W$ is a sub representation of $V$, its orthogonal complement

$$W^\perp = \{v \in V \colon \langle v, w \rangle = 0 \ \text{ for all } w \in W\}$$

is also a subrepresentation. Indeed, for $v \in W^\perp$ and $g \in G$, we have

$$\langle gv, w \rangle = \langle g^{-1}gv, g^{-1}w \rangle = \langle v, g^{-1}w \rangle = 0$$

for all $w \in W$. The invariant inner product descends to the corresponding representation of $\mathfrak{g}$, where condition (1) becomes

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle e^{tX} v, e^{tX} w \rangle = 0$$

for $X \in \mathfrak{g}$. In other words,

$$\langle Xv, w \rangle + \langle v, Xw \rangle = 0.$$

Again it can be checked easily that if $W$ is $\mathfrak{g}$-subrepresentation of $V$, then $W^\perp$ is also a $\mathfrak{g}$-subrepresentation.

## 4. Representations of $\mathfrak{sl}(2,\mathbb{C})$

**Theorem 4.1.** *Any finite dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible.*

*Proof (sketch).* From Section 3 we know that representations of $\mathfrak{sl}(2,\mathbb{C})$ are the same as representations of $\mathsf{SU}(2)$. If a finite dimensional representation $V$ of $\mathsf{SU}(2)$ is equipped with an invariant inner product, then one can easily show that $V$ is completely reducible by an induction argument. Hence it suffices to prove that any finite dimensional representation $V$ of $\mathsf{SU}(2)$ can be equipped with an invariant inner product.

Recall that $\mathsf{SU}(2)$ is homeomorphic to $\mathbb{S}^3$ and so is compact. In general, given a compact Lie group $G$ and a finite dimensional representation $V$ of $G$ equipped with an arbitrary inner product $\langle\cdot,\cdot\rangle$, one can always obtain an invariant inner product $\langle\cdot,\cdot\rangle_G$ on $V$ as follows (this is known as the *Weyl Unitary Trick*). First one can define a non-zero measure $\mu$ on the Borel $\sigma$-algebra of $G$ such that: (1) $\mu$ is locally finite, i.e. every point in $G$ has a neighbourhood with finite measure and (2) it is invariant under right-translation, i.e. $\mu(Eg)=\mu(E)$ for any $g\in G$ and any Borel set $E\subset G$. This is known as a *Haar measure*. Then one obtain a new inner product:

$$\langle v_1,v_2\rangle_G=\int_G\langle g\cdot v_1,g\cdot v_2\rangle\,\mathrm{d}\mu(g).$$

To see that $\langle\cdot,\cdot\rangle_G$ is indeed invariant, note that

$$\begin{aligned}\langle h\cdot v_1,h\cdot v_2\rangle_G&=\int_G\langle (gh)\cdot v_1,(gh)\cdot v_2\rangle\;\mathrm{d}\mu(g)\\&=\int_G\langle k\cdot v_1,k\cdot v_2\rangle\;\mathrm{d}\mu(kh^{-1})\quad\text{(make the substitution } k=gh)\\&=\int_G\langle k\cdot v_1,k\cdot v_2\rangle\;\mathrm{d}\mu(k)\quad\text{(because } \mu \text{ is right invariant)}\\&=\langle v_1,v_2\rangle_G,\end{aligned}$$

as required. For a completely algebraic proof of this result, without referring to Lie groups, the readers can consult [Kir08, **Section 6.3**]. □

**Corollary 4.2.** *Every finite dimensional representation $V$ of $\mathfrak{sl}(2,\mathbb{C})$ is the direct sum of its weight spaces. In other words,*

$$V=\bigoplus_{n\in\mathbb{Z}}V[n],$$

*where only finitely many $V[n]$ are non-zero.*

*Proof.* This follows directly from Corollary 1.4 and Theorem 4.1. □

Now we introduce a concept which will be useful in the next section. Let $V$ be a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, we define the ***formal character*** of $V$ to be

$$\mathrm{ch}(V):=\sum_{n\in\mathbb{Z}}\dim(V[n])t^n.$$

Note that $\mathrm{ch}(V)\in\mathbb{Z}[t,t^{-1}]$, the ring of Laurent polynomials in $t$ with integer coefficients, because $V$ is finite-dimensional. In particular, the character of the zero vector space is 0. From now on, we assume all representations of $\mathfrak{sl}(2,\mathbb{C})$ are finite-dimensional.

**Lemma 4.3.** *If $V$ and $W$ are representations of $\mathfrak{sl}(2,\mathbb{C})$, then*

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W).$$

*Proof.* We have

$$V \oplus W = \bigoplus_{n\in\mathbb{Z}} V[n] \oplus \bigoplus_{m\in\mathbb{Z}} W[m] = \bigoplus_{n\in\mathbb{Z}} V[n] \oplus W[n],$$

Taking formal character we obtain

$$\begin{aligned}\mathrm{ch}(V \oplus W) &= \sum_{n\in\mathbb{Z}} \dim((V\oplus W)[n])t^n \\ &= \sum_{n\in\mathbb{Z}} \dim(V[n] \oplus W[n])t^n \\ &= \sum_{n\in\mathbb{Z}} (\dim(V[n]) + \dim(W[n]))t^n \\ &= \mathrm{ch}(V) + \mathrm{ch}(W),\end{aligned}$$

as required. □

Our main result concerning formal characters is the following theorem.

**Theorem 4.4.** *Two representations $V$ and $W$ of $\mathfrak{sl}(2,\mathbb{C})$ are isomorphic if and only if $\mathrm{ch}(V) = \mathrm{ch}(W)$.*

*Proof.* The only if direction is clear from the definition, so we only need to prove the if direction. We proceed by induction on the dimension of $V$ and $W$. Notice that the condition $\mathrm{ch}(V) = \mathrm{ch}(W)$ implies that $V$ and $W$ have the same weight space decomposition and so they have the same dimension. Let $n$ be the dimension of $V$ and $W$. The case $n = 0$ is vacuously true, so consider $n > 0$. Let $\lambda$ be a highest weight of both $V$ and $W$. Then part (3) of Theorem 1.3 implies that $V$ contains a subrepresentation $V'$ isomorphic to $V_\lambda$ and $W$ contains a subrepresentation $W'$ isomorphic to $V_\lambda$. Now decompose

$$V = V' \oplus (V')^\perp, \quad W = W' \oplus (W')^\perp.$$

Here we assume that $V$ and $W$ are each equipped with an invariant inner product (see Section 3). Thus $(V')^\perp$ and $(W')^\perp$ are both subrepresentations. Now by assumption

$$\mathrm{ch}(V) = \mathrm{ch}(V') + \mathrm{ch}((V')^\perp) = \mathrm{ch}(W') + \mathrm{ch}((W')^\perp) = \mathrm{ch}(W).$$

But $\mathrm{ch}(V') = \mathrm{ch}(W') = \mathrm{ch}(V_\lambda)$, so $\mathrm{ch}((V')^\perp) = \mathrm{ch}((W')^\perp)$. Since $(V')^\perp$ ($(W')^\perp$ resp.) has dimension less than the dimension of $V$ ($W$ resp.), the induction hypothesis yields $(V')^\perp$ and $(W')^\perp$ are isomorphic as representations. Thus the representations $V$ and $W$ are isomorphic and that completes the induction. □

## 5. Clebsch-Gordan Rule

In this section we describe how to decompose the tensor product of two irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ into irreducibles. In general, given two representations $V$ and $W$ of a Lie algebra $\mathfrak{g}$ we can turn $V \otimes W$ into a representation as follows:

$$X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w, \quad X \in \mathfrak{g},\ v \in V,\ w \in W.$$

This action will look more natural on the level of Lie groups. Suppose $G$ is a Lie group whose Lie algebra is $\mathfrak{g}$ and $V$ and $W$ are representations of $G$. Then we can define an action of $G$ on $V \otimes W$ by

$$g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w, \quad g \in G,\ v \in V,\ w \in W.$$

Now for $X \in \mathfrak{g}$, its action on $v \otimes w \in V \otimes W$ is given by

$$X \cdot (v \otimes w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e^{tX} \cdot v \otimes e^{tX} \cdot w) = X \cdot v \otimes w + v \otimes X \cdot w,$$

as expected.

**Lemma 5.1.** *If $V$ and $W$ are representations of $\mathfrak{sl}(2,\mathbb{C})$, then*

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W).$$

*Proof.* Recall that $V$ and $W$ have bases consisting of eigenvectors of $h$. If $v \in V[n]$ and $w \in W[m]$, then

$$h \cdot (v \otimes w) = h \cdot v \otimes w + v \otimes h \cdot w = nv \otimes w + v \otimes mw = (n+m)v \otimes w.$$

It follows that

$$(V \otimes W)[k] = \bigoplus_{n+m=k} V[n] \otimes W[m].$$

Hence,

$$\begin{aligned}\mathrm{ch}(V \otimes W) &= \sum_{k\in\mathbb{Z}} \dim((V \otimes W)[k])t^k \\ &= \sum_{k\in\mathbb{Z}} \sum_{n+m=k} \dim(V[n]) \dim(W[m])t^k \\ &= \sum_{n\in\mathbb{Z}} \dim(V[n])t^n \sum_{m\in\mathbb{Z}} \dim(W[m])t^m \\ &= \mathrm{ch}(V)\mathrm{ch}(W),\end{aligned}$$

as required. □

**Theorem 5.2** (Clebsch-Gordan rule)**.** *Let $V_m$ and $V_n$ be the irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ of highest weights $m$ and $n$, respectively. Assume $m \geq n$, then*

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n+2} \oplus V_{m-n},$$

*as representations of $\mathfrak{sl}(2,\mathbb{C})$.*

*Proof.* By Theorem 4.4, it suffices to show that the two sides have the same formal characters. From Corollary 1.4, we have

$$\mathrm{ch}(V_n) = t^{-n} + t^{-n+2} + \cdots + t^{n-2} + t^n = \frac{t^{n+2} - t^{-n}}{t^2 - 1}.$$

So the formal character of the right hand side is given by b

$$\frac{t^{m+n+2} - t^{-m-n}}{t^2-1} + \frac{t^{m+n} - t^{-m-n+2}}{t^2-1} + \cdots + \frac{t^{m-n+4} - t^{-m+n-2}}{t^2-1} + \frac{t^{m-n+2} - t^{-m+n}}{t^2-1}. \tag{2}$$

Now note that

$$\begin{aligned}
& -t^{-m-n} - t^{-m-n+2} - \cdots - t^{-m+n-2} - t^{-m+n} \\
&= t^{-m}(t^{-n} + t^{-n+2} + \cdots + t^{n-2} + t^{n}) \\
&= t^{-m}\mathrm{ch}(V_n).
\end{aligned}$$

Also,

$$\begin{aligned}
& t^{m+n+2} + t^{m+n} + \cdots + t^{m-n+4} + t^{m-n+2} \\
&= t^{m+2}(t^{n} + t^{n-2} + \cdots + t^{-n+2} + t^{-n}) \\
&= t^{m+2}\mathrm{ch}(V_n).
\end{aligned}$$

Therefore (2) becomes

$$\mathrm{ch}(V_n)\left(\frac{t^{m+2} - t^{-m}}{t^2 - 1}\right) = \mathrm{ch}(V_n)\mathrm{ch}(V_m) = \mathrm{ch}(V_n \otimes V_m),$$

where the last equality follows from Lemma 5.1. $\square$

**Example.** When $m = n = 1$, we have

$$V_1 \otimes V_1 \cong V_2 \oplus V_0 \cong V_2 \oplus \mathbb{C},$$

where one can check easily that $V_2$ is the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$. The isomorphism is $v_0 \mapsto e$, $v_1 \mapsto h$ and $v_2 \mapsto f$.

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