PROJECT REPORT

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Contents

1 Preliminaries

1.1 Inner Product Spaces

This chapter provides a brief overview of knowledge needed for subsequent chapters. Most of the materials are taken from [5].

1.1.1 Inner Product

Let V be a finite-dimensional vector space over a field $\mathbb{F}$, where $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$. An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle$ in $\mathbb{F}$ and satisfies the following properties:

1. $\langle v, v\rangle \geq 0$ for all $v \in V$;
2. $\langle v, v\rangle = 0$ if and only if $v = 0$;
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;
4. $\langle av, w\rangle = a\langle v, w\rangle$ for all $a \in \mathbb{F}$ and $v, w \in V$;
5. $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$.

Conditions 3 and 4 can be combined into the requirement of linearity in the first slot. Thus an inner product is linear in the first slot and conjugate linear in the second slot, i.e.,

$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle \text{ and } \langle u, aw\rangle = \bar{a}\langle u, w\rangle .$$

Note that in the physics literature people often adopt the condition of linearity in the second slot and conjugate linearity in the first slot.

1.1.2 Norm

Let V be a nonzero finite-dimensional vector space. A *norm* on V is a function $\|\cdot\|: V \to \mathbb{R}$ satisfying the following properties

1. $\|v\| \geq 0$ for $v \in V$,
2. $\|v\| = 0$ if and only if $v = 0$,
3. $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$ (triangle inequality),
4. $\|\alpha v\| = |\alpha|\|v\|$ for $\alpha \in \mathbb{C}$, $v \in V$.

A vector space equipped with a norm is called a *normed vector space*. Given a normed vector space V, we can put a metric on V by

$$d(v, w) = \|v - w\|, \quad v, w \in V.$$

It's easy to verify that all the properties of a metric are satisfied. In particular,

$$d(v, w) = \|v - w\| = \| - (w - v)\| = | - 1|\|w - v\| = d(w, v).$$

Given an inner product on V, we can define a norm on V as follows

$$\|v\| = \sqrt{\langle v, v\rangle}, \quad v \in V.$$

Properties (1), (2), (4) of a norm follow directly from properties of an inner product. We check the triangle inequality:

$$\begin{aligned}
\|v + w\|^2 &= \langle v + w, v + w\rangle = \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2 \\
&\leq \|v\|^2 + 2| \langle v, w\rangle | + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\
&= (\|v\| + \|w\|)^2 ,
\end{aligned}$$

where the last inequality follows from the Cauchy–Schwarz inequality:

$$| \langle v, w\rangle | \leq \|v\|\|w\|,$$

which is true in any inner-product space. So every inner product induces a norm on a vector space. However, not every norm arises from an inner product. The most familiar norm on $\mathbb{R}^n$ is the *Euclidean norm*, given by

$$\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2 \text{ for all } v \in \mathbb{R}^n.}$$

It's easily seen that the Euclidean norm arises from the usual inner product on $\mathbb{R}^n$:

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i \text{ for all } v, w \in \mathbb{R}^n.$$

Another important norm on $\mathbb{R}^n$ is known as the *sup norm*, which is defined as

$$\|v\|_{\text{sup}} = \max \{|v_1|, |v_2|, ..., |v_n|\} .$$

Note that since

$$|v_i| \leq \sqrt{\sum_{i=1}^{n} v_i^2} \leq \sqrt{n} \max \{|v_1|, ..., |v_n|\} ,$$

the relationship between the sup norm and the Euclidean norm is given by

$$\|v\|_{\sup} \leq \|v\| \leq \sqrt{n}\|v\|_{\sup}.$$

The sup norm on $\mathbb{R}^2$ does not arise from any inner product on $\mathbb{R}^2$. If it did, then the following identity

$$\|u+v\|_{\sup}^2 + \|u-v\|_{\sup}^2 = 2\left(\|v\|_{\sup}^2 + \|w\|_{\sup}^2\right) \text{ for all } v, w \in \mathbb{R}^2,$$

known as the parallelogram law would hold (this can be verified simply by expanding the left hand side using the definition of norm induced from inner product). In the case where $u = (1,0)^T$ and $v = (0,1)^T$, it's easily seen that the above equality breaks down. Thus not every norm on V arises from an inner product on V.

1.1.3 Gram-Schmidt Procedure

Let v be a nonzero vector in V. For any vector $u \in V$, we'd like to write u as a sum of a scalar multiple of v and a vector orthogonal to v. Suppose

$$u = av + (u - av) \text{ for some scalar } a \in \mathbb{F}.$$

We want to choose a in such a way that

$$0 = \langle u - av, v\rangle = \langle u, v\rangle - a\langle v, v\rangle = \langle u, v\rangle - a\|v\|^2.$$

Since $v \neq 0$, we obtain

$$a = \frac{\langle u, v\rangle}{\|v\|^2}.$$

Thus

$$u = \frac{\langle u, v\rangle}{\|v\|^2}v + w,$$

where w is orthogonal to v. The vector

$$\frac{\langle u, v\rangle}{\|v\|^2}v$$

is called the *orthogonal projection* of u onto v. Recall that a list of vectors $(v_1, ..., v_n)$ is *orthonormal* if

$$\langle v_i, v_j\rangle = \delta_{ij} \text{ for } 1 \leq i, j \leq n.$$

The *Gram-Schmidt procedure* allows us to turn an independent list of vectors to an orthonormal list. More precisely,

Theorem 1.1 (Gram-Schmidt). *If $(v_1, ..., v_m)$ is an independent list of vectors, then there exists an orthonormal list of vectors $(e_1, ..., e_m)$ such that*

$$\operatorname{span}(v_1, ..., v_j) = \operatorname{span}(e_1, ..., e_j)$$

for $j = 1, ..., m$.

We sketch the construction below. First we construct an orthogonal list, and then turn it into an orthonormal list by the normalization $v/\|v\|$. Put $e_1 = v_1$ and

$$e_2 = v_2 - \frac{\langle v_2, e_1 \rangle}{\|e_1\|^2} e_1.$$

Generally,

$$e_j = v_j - \frac{\langle v_j, e_1 \rangle}{\|e_1\|^2} e_1 - \cdots - \frac{\langle v_j, e_{j-1} \rangle}{\|e_{j-1}\|^2} e_{j-1},$$

for $1 < j \leq m$. In other words, to obtain e_j, we subtract off from v_j its projection onto the subspace spanned by $(e_1, ..., e_{j-1})$. It can be shown that $(e_1, ..., e_m)$ is orthogonal and

$$\operatorname{span}(v_1, ..., v_j) = \operatorname{span}(e_1, ..., e_j)$$

for $j = 1, ..., m$. Thus every finite-dimensional inner-product space has an orthonormal basis. One important corollary of the Gram-Schmidt procedure is the following

Corollary 1.1. *Every orthonormal list of vectors can be extended to an orthonormal basis of V.*

To see why, suppose $(e_1, ..., e_m)$ is an orthonormal list of vectors. Then it is independent and can be extended to a basis of V, say

$$(e_1, ..., e_m, v_1, ..., v_n).$$

Apply the Gram-Schmidt procedure to the above list of vectors, we obtain an orthonormal basis of V. However, since the the first m vectors are orthonormal, they remain the same after the procedure. We've extended $(e_1, ..., e_m)$ to an orthonormal basis of V. $\square$

One of the reasons orthonormal bases are useful is because the elements of V can be expressed in a simple form. If $(e_1, ..., e_n)$ is an orthonormal basis for V and $v \in V$, then

$$v = a_1 e_1 + \cdots + a_n e_n \text{ for some scalars } a_1, ..., a_n \in \mathbb{F}.$$

Now since $(e_1, ..., e_n)$ is orthonormal,

$$\langle v, e_j \rangle = \left\langle \sum_{i=1}^{n} a_i e_i, e_j \right\rangle = \sum_{i=1}^{n} a_i \langle e_i, e_j \rangle = a_j$$

for $j = 1, ..., n$. Thus

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n.$$

We can also obtain the norm of v:

$$\|v\|^2 = \langle v, v \rangle = \left\langle \sum_{i=1}^{n} \langle v, e_i \rangle e_i, \sum_{j=1}^{n} \langle v, e_j \rangle e_j \right\rangle = \sum_{i=1}^{n} | \langle v, e_i \rangle |^2.$$

1.1.4 Orthogonal Projections

If U is a subset of V, recall that the *orthogonal complement* of U, denoted by $U^{\perp}$ is

$$U^{\perp} = \{v \in V\colon\ \langle v, u\rangle = 0 \quad \text{for all } u \in U\}\,.$$

The next theorem shows that every subspace of an inner-product space leads to a natural direct sum decomposition of the whole space.

Theorem 1.2. *If U is a subspace of V, then*

$$V = U \oplus U^{\perp}.$$

Proof. First we show that every element v of V can be written as

$$v = u + w,$$

where $u \in U$ and $w \in U^{\perp}$. Let $(u_1, ..., u_m)$ be an orthonormal basis for U and put

$$u = \langle v, u_1\rangle\, u_1 + \langle v, u_2\rangle\, u_2 + \cdots + \langle v, u_m\rangle\, u_m.$$

Then clearly $u \in U$. Now we write v as

$$v = u + (v - u).$$

We need to show that $v - u \in U^{\perp}$. Indeed,

$$\langle v - u, u_j\rangle = \langle v, u_j\rangle - \langle u, u_j\rangle = \langle v, u_j\rangle - \langle v, u_j\rangle = 0$$

for $j = 1, ..., m$. Thus $v - u$ is orthogonal to each element in a basis of U and so it is orthogonal to every element of U.

Now take $z \in U \cap U^{\perp}$, then

$$\langle z, z\rangle = 0.$$

Hence $z = 0$ and the intersection of U and $U^{\perp}$ is trivial. $\square$

1.2 Linear Functionals and Adjoints

1.2.1 The Adjoint of an Operator

Recall that a *linear functional* on V is a linear map from V to the scalar $\mathbb{F}$. For instance, if w is any element of V, the map $\varphi\colon\ V \to \mathbb{F}$ given by

$$\varphi(v) = \langle v, w\rangle \ \text{ for all } v \in V$$

is a linear functional due to properties of the inner product. Interestingly, it turns out that every linear functional is of this form.

Theorem 1.3. *Suppose φ is a linear functional on V. Then there is a unique $v \in V$ such that*

$$\varphi(u) = \langle u, v \rangle$$

for every $u \in V$.

Proof. We first show the existence of v. Let $(e_1, ..., e_n)$ be an orthonormal basis for V. For any $u \in V$, it can be written as

$$u = \langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n.$$

Thus

$$\begin{aligned} \varphi(u) &= \varphi\left(\langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n\right) \\ &= \langle u, e_1 \rangle \varphi(e_1) + \cdots + \langle u, e_n \rangle \varphi(e_n) \text{ (by linearity of } \varphi) \\ &= \left\langle u, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \right\rangle \text{ (because } \varphi(e_i) \in \mathbb{F}) . \end{aligned}$$

So v can be chosen as

$$v = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n.$$

For the uniqueness of v, assume that

$$\langle u, v_1 \rangle = \langle u, v_2 \rangle \text{ for all } u \in V.$$

Then

$$\langle u, v_1 - v_2 \rangle = 0 \text{ for all } u \in V.$$

Choose u to be $v_1 - v_2$, we obtain $\|v_1 - v_2\| = 0$. Therefore, $v_1 = v_2$. □

We are now ready to define the adjoint of an operator. Let W be a finite-dimensional inner-product space and $T\colon V \to W$ be a linear map. The *adjoint* of T, denoted by T^* is a linear map $W \to V$ and is defined as follows. For any $w \in W$, the map $\varphi\colon V \to \mathbb{F}$ given by

$$\varphi(u) = \langle T(u), w \rangle \text{ for all } u \in V$$

is a linear functional on V since T is linear (note that here we use the inner product on W). Thus there exists a unique $v \in V$ such that

$$\varphi(u) = \langle T(u), w \rangle = \langle u, v \rangle \text{ for all } u \in V.$$

We define $T^*(w) = v$. The above equality can be written as

$$\langle T(u), w \rangle = \langle u, T^*(w) \rangle \text{ for all } u \in V \text{ and } w \in W.$$

We should check that T^* is linear. Indeed, for any $u \in V$ and $w_1, w_2 \in W$,

$$\begin{aligned} \langle u, T^*(w_1 + w_2) \rangle &= \langle T(u), w_1 + w_2 \rangle = \langle T(u), w_1 \rangle + \langle T(u), w_2 \rangle \\ &= \langle u, T^*(w_1) \rangle + \langle u, T^*(w_2) \rangle = \langle u, T^*(w_1) + T^*(w_2) \rangle . \end{aligned}$$

We conclude that $T^*(w_1 + w_2) = T^*(w_1) + T^*(w_2)$. Similarly, for any $a \in \mathbb{F}$,

$$\begin{aligned} \langle u, T^*(aw) \rangle &= \langle T(u), aw \rangle = \overline{a} \langle T(u), w \rangle \\ &= \overline{a} \langle u, T^*(w) \rangle = \langle u, aT^*(w) \rangle . \end{aligned}$$

We conclude that $T^*(aw) = aT^*(w)$. Thus T^* is linear.

Let's try to find the adjoint of a simple operator. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1)$$

To find the adjoint, consider for any $(y_1, y_2) \in \mathbb{R}^2$

$$\begin{aligned} \langle T(x_1, x_2, x_3), (y_1, y_2) \rangle &= \langle (x_2 + 3x_3, 2x_1), (y_1, y_2) \rangle = x_2 y_1 + 2x_1 y_2 + 3x_3 y_1 \\ &= \langle (x_1, x_2, x_3), (2y_2, y_1, 3y_1) \rangle . \end{aligned}$$

Thus

$$T^*(y_1, y_2) = (2y_2, y_1, 3y_1).$$

We list some simple properties of the adjoint

Proposition 1.1. *The function $T \mapsto T^*$ satisfies*

1. $(S + T)^* = S^* + T^*$ *(additivity);*
2. $(aT)^* = \bar{a}T^*$ *for $a \in \mathbb{F}$ (conjugate homogeneity);*
3. $(T^*)^* = T$ *(adjoint of adjoint);*
4. $I^* = I$ *(identity)*
5. $(ST)^* = T^*S^*$ *(product).*

Proof. These properties follow from the definition of the adjoint. For instance, property 3 holds because

$$\langle x, (T^*)^* y \rangle = \langle T^* x, y \rangle = \overline{\langle y, T^* x \rangle} = \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle .$$

And for property 5, note that

$$\langle x, (ST)^* y \rangle = \langle STx, y \rangle = \langle Tx, S^* y \rangle = \langle x, T^*(S^* y) \rangle .$$

□

Recall that the conjugate transpose of a matrix A, denoted by A^* is the matrix obtained from A by first taking the transpose and then take the conjugation of each entry. In symbols,

$$(A^*)_{ij} = \overline{A_{ji}}, \text{ for all } i, j.$$

The notation is no coincidence. We'll show that the matrix representation of the adjoint of an operator with respect to some orthonormal bases is the conjugate transpose of that of the operator.

Proposition 1.2. *Suppose $T\colon V \to W$ is a linear map. If $(e_1, ..., e_n)$ is an orthonormal basis for V and $(f_1, ..., f_m)$ is an orthonormal basis for W, then*

$$M(T^*, (f_1, ..., f_m), (e_1, ..., e_n))$$

is the conjugate transpose of

$$M(T, (e_1, ..., e_n), (f_1, ..., f_m)).$$

Proof. To avoid cumbersome notations, we'll use $M(T^*)$ and $M(T)$. The bases are clear from context. Note that the (i, j)-entry of $M(T)$ is given by

$$(M(T))_{ij} = \langle T(e_j), f_i \rangle = \langle e_j, T^*(f_i) \rangle = \overline{\langle T^*(f_i), e_j \rangle} = \overline{(M(T^*))_{ji}},$$

where the first and last equalities follow from the orthonormality of the bases and the second equality follows from the definition of adjoint. By definition of conjugate transpose,

$$M(T)^* = M(T^*).$$

□

1.2.2 Self-adjoint Operators

An operator T on some finite-dimensional inner-product space is called *self-adjoint* or *Hermitian* if $T^* = T$. From properties of the adjoint, it follows that the sum of two self-adjoint operators is self-adjoint and the product of a real scalar and a self-adjoint operator is self-adjoint. Self-adjoint operators have several nice properties which we're going to illustrate below.

Proposition 1.3. *Every eigenvalue of a self-adjoint operator is real.*

Proof. Suppose T is a self-adjoint operator on V and λ is an eigenvalue of T. Then there exists some nonzero vector $v \in V$ such that $T(v) = \lambda v$. We have

$$\begin{aligned} \lambda \|v\|^2 &= \lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle T(v), v \rangle = \langle v, T^* v \rangle \\ &= \langle v, T(v) \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \langle v, v \rangle = \overline{\lambda} \|v\|^2. \end{aligned}$$

We conclude that $\lambda = \overline{\lambda}$. Thus λ is real. □

The next result is only true for complex inner-product spaces.

Proposition 1.4. *If V is a complex inner-product space and T is an operator on V such that*

$$\langle Tv, v \rangle = 0$$

for all $v \in V$, then $T = 0$.

This property is not true for real inner-product space. Consider the rotation of $\mathbb{R}^2$ by $\pi/2$ counterclockwise. Clearly, $\langle Tv, v \rangle = 0$, but $T \neq 0$.

Proof. The key is the following equality

$$\langle Tu, w\rangle = \frac{\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle}{4} + \frac{\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle}{4} i,$$

which can be verified by expanding the right hand side. Since $\langle Tv, v\rangle = 0$ for all $v \in V$, we conclude that $\langle Tu, w\rangle = 0$ for all $u.w \in V$. In particular, $\langle Tu, Tu\rangle = 0$ for all $u \in V$, which implies that $T = 0$. □

One important corollary of the above proposition is the following.

Corollary 1.2. *Let V be a complex inner-product space and T be a linear operator on V. Then T is self-adjoint if and only if*

$$\langle Tv, v\rangle \in \mathbb{R}$$

for every $v \in V$.

Proof. Let $v \in V$. Then

$$\begin{aligned} \langle Tv, v\rangle - \overline{\langle Tv, v\rangle} &= \langle Tv, v\rangle - \langle v, Tv\rangle \\ &= \langle Tv, v\rangle - \langle T^*v, v\rangle \\ &= \langle (T - T^*)v, v\rangle . \end{aligned}$$

Thus the condition $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$ is equivalent to $\langle (T - T^*)v, v\rangle = 0$ for all $v \in V$. Since we are over the complex field, the latter condition is equivalent to $T - T^* = 0$ by Proposition 1.4, i.e., $T = T^*$. □

We remark that an $n \times n$ matrix A can be thought of as an operator $\mathbb{C}^n \to \mathbb{C}^n$ in the usual manner. The (i, j)-entry of A is given by

$$A_{ij} = \langle Ae_j, e_i\rangle ,$$

where e_i is a standard basis element for $\mathbb{C}^n$ and the (j, i)-entry of A^*, the conjugate transpose of A is given by

$$A^*_{ji} = \langle A^*e_i, e_j\rangle = \overline{A_{ij}} = \langle e_i, Ae_j\rangle .$$

Thus if $A^* = A$, we have

$$\langle Ae_i, e_j\rangle = \langle e_i, Ae_j\rangle \quad \text{for all } i, j.$$

By properties of inner product,

$$\langle Av, w\rangle = \langle v, Aw\rangle \quad \text{for all } v, w \in \mathbb{C}^n.$$

Hence the matrix A is self-adjoint if and only if $A^* = A$, as expected.

1.2.3 The Spectral Theorem

One of the nicest properties of complex self-adjoint operators is there exists an orthonormal basis for V consisting of eigenvectors of the operator. This result is known as the spectral theorem and it is our aim in this subsection to establish it. We record the theorem below.

Theorem 1.4 (Complex Spectral Theorem)**.** *Suppose that V is a complex inner-product space and T is an operator on V. Then T is self-adjoint if and only if V has an orthonormal basis consisting of eigenvectors of T and all eigenvalues of T are real.*

To prove the theorem, we need several lemmas. The most important one is

Lemma 1.1. *Suppose V is a complex vector space and T is an operator on V. Then T has an upper-triangular matrix with respect to some basis of V.*

Proof. We're going to prove the result using induction on the dimension of V. The case where $\dim V = 1$ is clearly true. So we assume that $\dim V > 1$ and the lemma holds true for every complex vector space of dimension less than that of V and bigger than 0.

Let T be an operator on V. Since V is a complex vector space, T has an eigenvalue λ (the characteristic polynomial of T always has a solution). Consider the following subspace of V

$$U = \mathrm{range}\,(T - \lambda I).$$

Because λ is an eigenvalue of T, there exists some nonzero vector $v \in V$ such that $(T - \lambda I)v = 0$. Thus $T - \lambda I$ is not surjective and so $\dim U < \dim V$. Moreover, for every $u \in U$, we can write

$$T(u) = (T - \lambda I)u + \lambda u.$$

Clearly $(T - \lambda I)u \in U$ and $\lambda u \in U$. Thus U is invariant under T. We can now apply the induction hypothesis to $(U, T|_U)$. Consequently there exists a basis $(u_1, ..., u_m)$ for U with respect to which $T|_U$ has an upper triangular matrix. Extend that basis to a basis for V

$$(u_1, ..., u_m, v_1, ..., v_n).$$

Now we have

$$T(u_i) = T|_U\,(u_i) \in \mathrm{span}(u_1, ..., u_i)$$

for $i = 1, ..., m$ because the matrix of $T|_U$ is upper-triangular. For the v_k, we can write Tv_k as

$$T(v_k) = (T - \lambda I)v_k + \lambda v_k$$

for $k = 1, ..., n$. Note that $(T - \lambda I)v_k \in U$ and hence it belongs to the span of $(u_1, ..., u_m)$. We conclude that $T(v_k) \in \mathrm{span}(u_1, ..., u_m, v_1, ..., v_k)$ for $k = 1, ..., n$. We've found a required basis. □

The next lemma tells us that in fact we can choose the basis to be orthonormal.

Lemma 1.2. *Suppose V is a complex vector space and T is an operator on V. Then T has an upper-triangular matrix with respect to some* orthonormal *basis of V.*

Proof. We know that there exists a basis $(v_1, ..., v_n)$ for V with respect to which T is upper-triangular. In other words, the subspaces $(v_1, ...v_i)$ are invariant under T for $i = 1, ..., n$. Apply the Gram-Schmidt procedure to $(v_1, ..., v_n)$ we obtain an orthonormal basis $(e_1, ..., e_n)$ for V. Note that the procedure doesn't change the span of the vectors, hence

$$\text{span}(v_1, ..., v_i) = \text{span}(e_1, ..., e_i) \text{ for } i = 1, ..., n.$$

Thus $\text{span}(e_1, ..., e_i)$ are also invariant under T for $i = 1, ..., n$, which implies that T is upper-triangular with respect to $(e_1, ..., e_n)$. □

We are now ready to prove the complex spectral theorem. Suppose T is self-adjoint and let $(e_1, ..., e_n)$ be an orthonormal basis for V such that $M(T)$ is upper-triangular (again we omit the basis, it is understood to be $(e_1, ..., e_n)$). By orthonormality of $(e_1, ..., e_n)$, $M(T^*)$ is the conjugate transpose of $M(T)$, i.e.,

$$M(T^*)_{ij} = \overline{M(T)_{ji}} \text{ for } 1 \leq i, j \leq n.$$

Since T is self-adjoint,

$$M(T)_{ij} = \overline{M(T)_{ji}} \text{ for } 1 \leq i, j \leq n.$$

Because T is upper-triangular, $M(T)_{ij} = 0$ for $n \geq i > j \geq 1$. For $i < j$, the above equation implies that $M(T)_{ij}$ is also 0. Thus $M(T)$ is in fact diagonal and the basis $(e_1, ..., e_n)$ is an orthonormal basis for V consisting of eigenvectors of T.

For the other direction, suppose that V has an orthonormal basis $(e_1, ..., e_n)$ consisting of eigenvectors of T and that

$$T(e_i) = \lambda_i e_i \text{ for } i = 1, ..., n,$$

where $\lambda_i \in \mathbb{R}$ for $i = 1, ..., n$. For any $v \in V$, we have

$$v = a_1 e_1 + \cdots + a_n e_n \text{ for } a_i \in \mathbb{C}.$$

Thus

$$\langle Tv, v \rangle = \left\langle \sum_{i=1}^{n} a_i T e_i, \sum_{j=1}^{n} a_j e_j \right\rangle = \left\langle \sum_{i=1}^{n} a_i \lambda_i e_i, \sum_{j=1}^{n} a_j e_j \right\rangle = \sum_{i=1}^{n} |a_i|^2 \lambda_i$$

by orthonormality of $(e_1, ..., e_n)$. Since all eigenvalues of T are real, we conclude that $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$. Hence T is self-adjoint by Corollary 1.2. □

1.2.4 Positive Operators

Let T be an operator on some complex finite-dimensional inner-product space V. Then T is called *positive* if

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$. For an example of a positive operator, take a nonzero vector $v \in V$ and consider the orthogonal projection onto v given by

$$P_v(u) = \frac{\langle u, v\rangle}{\|v\|^2} v \text{ for all } u \in V.$$

We have

$$\langle P_v(u), u\rangle = \left\langle \frac{\langle u, v\rangle}{\|v\|^2} v, u \right\rangle = \frac{|\langle u, v\rangle|^2}{\|v\|^2} \geq 0.$$

Thus P_v is positive. Another important characteristic of positive operators is the following.

Proposition 1.5. *An operator T is positive if and only if it is self-adjoint and all of its eigenvalues are non-negative.*

Proof. Suppose that T is positive. Since V is a complex vector space, the condition $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in V$ implies that T is self-adjoint. If λ is an eigenvalue of T with eigenvector v, then

$$0 \leq \langle Tv, v\rangle = \lambda \langle v, v\rangle.$$

Thus $\lambda \geq 0$ since $\langle v, v\rangle > 0$.

Now if T is self-adjoint and all the eigenvalues of T are non-negative. By the spectral theorem, there exists an orthonormal basis $(e_1, ..., e_n)$ for V consisting of eigenvectors of T. For any $v \in V$, we have

$$v = a_1 e_1 + \cdots + a_n e_n \text{ for } a_i \in \mathbb{C}.$$

Thus

$$\langle Tv, v\rangle = \left\langle \sum_{i=1}^n a_i T e_i, \sum_{j=1}^n a_j e_j \right\rangle = \left\langle \sum_{i=1}^n a_i \lambda_i e_i, \sum_{j=1}^n a_j e_j \right\rangle = \sum_{i=1}^n |a_i|^2 \lambda_i$$

by orthonormality of $(e_1, ..., e_n)$. Since each eigenvalue λ_i is non-negative, we conclude that

$$\langle Tv, v\rangle \geq 0$$

for all $v \in V$. $\square$

1.2.5 Functions of Self-adjoint Operators

One advantage of the spectral theorem is that it allows us to define operator-valued functions of self-adjoint operators in terms of their eigenvalues. Let A be a self-adjoint operator on V and $\beta = (v_1, ..., v_n)$ is an orthonormal basis for V consisting of eigenvectors of A, i.e.,

$$Av_i = \lambda_i v_i, \quad i = 1, ..., n.$$

If f is a complex-valued function that is defined on the eigenvalues of A, then we define $f(A)$ to be

$$f(A)v_i = f(\lambda_i)v_i, \quad i = 1, ..., n.$$

and extend to the whole V using linearity. To have a good definition, we need to show that it is independent of the choice of the orthonormal basis. Indeed, if w is an eigenvector of A corresponding to an eigenvalue λ, then w can be expressed as a linear combination of independent eigenvectors

$$w = \sum_j a_j v_j,$$

where each $v_j \in \beta$ is an eigenvector of A corresponding to the eigenvalue λ. It follows that

$$f(A)w = \sum_j a_j f(A)v_j = \sum_j a_j f(\lambda)v_j = f(\lambda)w.$$

Thus the definition of $f(A)$ doesn't depend on the choice of basis. From the definition, we also have

$$\begin{aligned}(f+g)(A) &= f(A) + g(A),\\ (fg)(A) &= f(A)g(A),\\ (f \circ g)(A) &= f(g(A)).\end{aligned}$$

Note that our definition doesn't account for every possible functions on operators. For instance, the following function

$$f(A) = PA$$

for some fixed matrix P is not obtained from any complex-valued function f. We give several important examples of operator-valued functions.

1. If f is a polynomial
$$f(z) = a_0 + a_1 z + \cdots + a_k z^k,$$
then $f(A)$ is given by
$$f(A) = a_0 + a_1 A + \cdots + a_k A^k.$$
Note that the left hand side is defined in terms of the eigenvalues of A and the right hand side is the usual operations of operators. It's easy to check that they're in fact the same.

2. If f is given in term of a power series
$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

and the series converges on the eigenvalues of A, then we write $f(A)$ as

$$f(A) = \sum_{k=0}^{\infty} a_k A^k.$$

We'll see later that this is just a special case of convergence of operators.

3. If A is a self-adjoint operator with non-zero eigenvalues, then
$$f(z) = z^{-1}$$
is well-defined on the eigenvalues of A. The function
$$f(A)v = \lambda^{-1}v$$
is the usual A^{-1}.

4. If B is a self-adjoint operator with non-negative eigenvalues, i.e., a positive operator, then
$$g(z) = \sqrt{z}$$
is well-defined on the eigenvalues of A (here we just consider the positive square root). The function
$$g(B)v = \sqrt{\lambda}v$$
is known as the *positive square root* of B, which we denote by $\sqrt{B}$. Clearly,
$$(\sqrt{B})^2 = B,$$
as expected.

1.3 Norms of Operators

1.3.1 Bounded Linear Opearators

Let V and W be inner-product spaces. A linear map $T\colon V \to W$ is said to be *bounded* if there exists a real number c such that

$$\|T(v)\| \leq c\|v\|$$

for every $v \in V$. Note that here $\|T(v)\|$ denotes the norm of $T(v)$ in W. When there's no ambiguity, we'll use the same notation $\|\cdot\|$ to denote the norm. Note that the concept of boundedness here only applies to linear maps and it is different from the usual definition of boundedness used in calculus. For instance, the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is unbounded in the usual sense of calculus since its range is the whole of $\mathbb{R}$. However, it is a bounded linear map since

$$\|f(x)\| = \|x\| \leq c\|x\|$$

for any real number $c > 1$. Bounded linear maps have a nice property, as given by the next proposition.

Proposition 1.6. *If $T\colon V \to W$ is a bounded linear map, then T is continuous.*

Proof. Since T is bounded, there exists $c > 0$ such that

$$\|T(v)\| \leq c\|v\|, \quad v \in V.$$

Let $\varepsilon > 0$ be arbitrary. For $v, w \in V$ such that $\|v - w\| < \varepsilon/c$, we have

$$\|T(v) - T(w)\| = \|T(v - w)\| \leq c\|v - w\| < \varepsilon,$$

where the first equality uses the linearity of T. Thus T is continuous. □

The next proposition shows that linear maps between finite-dimensional inner-product spaces are continuous, as expected.

Lemma 1.3. *Let $T\colon V \to W$ be a linear map between finite-dimensional inner-product spaces, then T is bounded.*

Proof. Let $\{v_1, ..., v_n\}$ be an orthonormal basis for V. For any $v \in V$, we have

$$v = \sum_{k=1}^{n} a_k v_k, \quad a_k \in \mathbb{C}.$$

Thus

$$\|v\|^2 = \langle v, v \rangle = \left\langle \sum_{k=1}^{n} a_k v_k, \sum_{l=1}^{n} a_l v_l \right\rangle = \sum_{k=1}^{n} |a_k|^2,$$

where the last equality follows from the orthonormality of $\{v_1, ..., v_n\}$. The Cauchy–Schwarz inequality now gives

$$\sum_{k=1}^{n} |a_k|^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} |a_k| \right)^2.$$

Hence we obtain

$$\|v\| \geq \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |a_k|.$$

Now

$$\|T(v)\| = \left\| \sum_{k=1}^{n} a_k T(v_k) \right\| \leq \sum_{k=1}^{n} |a_k| \|T(v_k)\| \leq c \sum_{k=1}^{n} |a_k|,$$

where $c > 0$ is the maximum value of $\{\|T(v_1)\|, ..., \|T(v_k)\|\}$, independent of the vector v. Therefore,

$$\|T(v)\| \leq c \sum_{k=1}^{n} |a_k| \leq c\sqrt{n}\|v\|$$

for any $v \in V$. So T is bounded. □

Recall that a subset K of a metric space M is compact if every open cover of K contains a finite subcover. Let V be an inner product space and $\{v_1, ..., v_n\}$ be an orthonormal basis for V. The following map $T\colon \mathbb{C}^n \to V$ given by

$$T(e_k) = v_k, \quad k = 1, ..., n,$$

where $\{e_1, ..., e_n\}$ is the standard basis for $\mathbb{C}^n$, is clearly an invertible linear map. Since linear maps are continuous, it is also an homeomorphism. Let x be any vector of $\mathbb{C}^n$ and suppose

$$x = \sum_{k=1}^{n} a_k e_k, \quad a_k \in \mathbb{C},$$

then

$$\begin{aligned}\|T(x)\|^2 &= \left\langle \sum_{k=1}^{n} a_k T(e_k), \sum_{l=1}^{n} a_l T(e_l) \right\rangle = \left\langle \sum_{k=1}^{n} a_k v_k, \sum_{l=1}^{n} a_l v_l \right\rangle \\ &= \sum_{k=1}^{n} |a_k|^2 = \left\langle \sum_{k=1}^{n} a_k e_k, \sum_{l=1}^{n} a_l e_l \right\rangle = \|x\|^2.\end{aligned}$$

So T preserves the norm. Therefore, the subset

$$\{v \in V\colon \|v\| = 1\}$$

of V is the image of the subset

$$\{x \in \mathbb{C}\colon \|x\| = 1\}$$

of $\mathbb{C}^n$ under T. Since the latter set is compact, we know that the former set is also compact.

1.3.2 The Operator Norm

Let $T\colon V \to W$ be a linear map between finite-dimensional inner-product spaces. Consider the following set

$$A = \left\{ \frac{\|T(v)\|}{\|v\|}\colon v \in V \text{ and } v \neq 0 \right\}.$$

(A is nonempty because we assume that V is nonzero.) Since T is bounded, the set A is bounded above by some $c > 0$. The *operator norm* of T is defined to be

$$\|T\| = \sup A.$$

The following inequality is immediate from the definition

$$\|T(v)\| \leq \|T\| \|v\| \text{ for all } v \in V.$$

We check that $\|\cdot\|$ is indeed a norm on the space of linear maps $V \to W$. Property (1) is trivial because each element of A is nonnegative. For property (2), if $T = 0$, then $A = \{0\}$, thus $\|T\| = 0$. Conversely, if $\|T\| = 0$, then

$$\|T(v)\| \leq \|T\|\|v\| = 0 \text{ for all } v \in V.$$

Thus $T = 0$. To show property (3), take any linear maps T and S and note that for any nonzero $v \in V$,

$$\begin{aligned}\|(T+S)(v)\| = \|T(v) + S(v)\| &\leq \|T(v)\| + \|S(v)\| \\ &\leq \|T\|\|v\| + \|S\|\|v\| = (\|T\| + \|S\|)\|v\|.\end{aligned}$$

Thus

$$\frac{\|(T+S)(v)\|}{\|v\|} \leq \|T\| + \|S\|$$

for any nonzero $v \in V$. It follows that

$$\|T + S\| \leq \|T\| + \|S\|.$$

Finally, for any $\alpha \in \mathbb{C}$,

$$\begin{aligned}\|\alpha T\| &= \sup\left\{\frac{\|\alpha T(v)\|}{\|v\|} : v \in V \text{ and } v \neq 0\right\} \\ &= \sup\left\{|\alpha|\frac{\|T(v)\|}{\|v\|} : v \in V \text{ and } v \neq 0\right\} \\ &= |\alpha| \sup\left\{\frac{\|T(v)\|}{\|v\|} : v \in V \text{ and } v \neq 0\right\} \\ &= |\alpha|\|T\|.\end{aligned}$$

That shows property (4).

It turns out that to find the operator norm, we don't need to consider every $v \in V$, as the next proposition shows.

Proposition 1.7. *Let $T\colon V \to W$ be a linear map between finite-dimensional inner-product spaces. An alternative formula for the operator norm of T is*

$$\|T\| = \sup\{\|T(v)\| : v \in V \textit{ and } \|v\| = 1\}.$$

Proof. We let b denote the right hand side of the above expression. The subset considered is clearly a subset of A (when $\|v\| = 1$), thus $b \leq \|T\|$. On the other hand, since T is linear,

$$\frac{\|T(v)\|}{\|v\|} = \left\|\frac{1}{\|v\|}T(v)\right\| = \left\|T\left(\frac{v}{\|v\|}\right)\right\| \leq b$$

for every nonzero $v \in V$ because the norm of $v/\|v\|$ is 1. Thus $\|T\| \leq b$. We conclude that $b = \|T\|$. □

Since $\{v \in V : \|v\| = 1\}$ is compact, we can in fact replace supremum by maximum in the formula for the operator norm. Some other useful properties of the operator norm are given in the next proposition.

Proposition 1.8. *Let T, S be operators on a finite-dimensional inner-product space V, then*

1. $\|TS\| \leq \|T\|\|S\|$,
2. $|\langle T(v), w\rangle| \leq \|T\|\|v\|\|w\|$ *for all* $v, w \in V$,
3. $\|T\| = \|T^*\|$,
4. $\|TT^*\| = \|T^*T\| = \|T\|^2$.

Proof. To prove (1), note that

$$\|TS(v)\| = \|T(S(v))\| \leq \|T\|\|S(v)\| \leq \|T\|\|S\|\|v\|$$

for all $v \in V$. Thus $\|TS\| \leq \|T\|\|S\|$. For (2), we apply the Cauchy-Schwarz inequality

$$|\langle T(v), w\rangle| \leq \|T(v)\|\|w\| \leq \|T\|\|v\|\|w\|.$$

Property (3) is trivial for the case $T = 0$. Assume $T \neq 0$, we have

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*Tv, v\rangle \leq \|T^*T\|\|v\|^2$$

for any $v \in V$. Here the last inequality follows from property (2). Taking the squared root, we get

$$\|T(v)\| \leq \sqrt{\|T^*T\|}\|v\| \text{ for any } v \in V.$$

Thus by definition of the operator norm,

$$\|T\| \leq \sqrt{\|T^*T\|}.$$

Hence,

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\|,$$

where the last inequality uses property (1). Dividing both sides by $\|T\| \neq 0$ yields

$$\|T\| \leq \|T^*\|.$$

The same inequality applies to T^*, thus

$$\|T^*\| \leq \|T^{**}\| = \|T\|.$$

We conclude that $\|T\| = \|T^*\|$. Finally, since $\|T\| = \|T^*\|$,

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

Thus $\|T\|^2 = \|T^*T\|$. Applying the same equality to T^* we obtain

$$\|T\|^2 = \|T^*\|^2 = \|T^{**}T^*\| = \|TT^*\|.$$

That completes the proof. ∎

1.3.3 The Operator Norm of Self-Adjoint Operators

So far we haven't given any examples of how to compute the operator norm. If $T = 0$, then $\|T\| = 0$. The simplest non-trivial example is the identity map $T(v) = v$ for all $v \in V$. In which case, it's easily seen that $\|T\| = 1$. For a general operator, it's not a trivial matter to compute the operator norm directly from the definition. In this subsection, we focus on finding the operator norm of self-adjoint operators. Our main result is the following.

Proposition 1.9. *Let T be a self-adjoint operator on a finite-dimensional inner-product space V. Then*

$$\|T\| = \max\{|\lambda| \colon \lambda \text{ is an eigenvalue of } T\}.$$

Actually this proposition allows us to find the operator norm of every linear operator T since we know that

$$\|TT^*\| = \|T^*T\| = \|T\|^2$$

and the operator TT^* is self-adjoint.

Proof. The proof becomes quite straightforward once we know the spectral theorem. Let $(e_1, ..., e_n)$ be an orthonormal basis for V consisting of eigenvectors of T, i.e.,

$$Te_i = \lambda_i e_i, \quad i = 1, ..., n.$$

Then for any $v \in V$ such that $\|v\| = 1$ we have

$$\begin{aligned}
\|Tv\|^2 &= \left\| T\left(\sum_{i=1}^n \langle v, e_i\rangle e_i\right)\right\|^2 = \left\|\sum_{i=1}^n \langle v, e_i\rangle Te_i\right\|^2 \\
&= \left\|\sum_{i=1}^n \langle v, e_i\rangle \lambda_i e_i\right\|^2 = \sum_{i=1}^n |\langle v, e_i\rangle \lambda_i|^2 \\
&\le \max_i |\lambda_i|^2 \sum_{i=1}^n |\langle v, e_i\rangle|^2 = \max_i |\lambda_i|^2
\end{aligned}$$

and the equality occurs when v is one of $(e_1, ..., e_n)$. Thus

$$\|T\| = \sup\{\|Tv\| \colon \|v\| = 1\} = \max_i |\lambda_i|.$$

$\square$

For an example, suppose we want to find the operator norm of the following self-adjoint operator $\mathbb{C}^2 \to \mathbb{C}^2$

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\lambda^2 - 3\lambda + 1 = 0,$$

which has solutions

$$\lambda = \frac{3 \pm \sqrt{5}}{2}.$$

Hence

$$\|A\| = \frac{3 + \sqrt{5}}{2}.$$

If we apply the definition directly, then we want to find the maximum of

$$2|u|^2 + 5|v|^2 + 3i(v\bar{u} - u\bar{v}),$$

where u and v are complex numbers subject to the condition $|u|^2 + |v|^2 = 1$, which is not very easy.

The next proposition gives yet another formula for the operator norm of a self-adjoint operator.

Proposition 1.10. *Let T be a self-adjoint operator on an inner-product space V. Then*

$$\|T\| = \sup \{ | \langle Tv, v \rangle | : \|v\| = 1 \}.$$

Proof. Since T is self-adjoint, we can choose an orthonormal basis for V consisting of eigenvectors of T, i.e.,

$$Te_i = \lambda_i e_i, \quad i = 1, ..., n.$$

For any $v \in V$ such that $\|v\| = 1$, we have

$$\begin{aligned} | \langle Tv, v \rangle | &= \left| \left\langle \sum_{i=1}^{n} \langle v, e_i \rangle \lambda_i e_i, \sum_{j=1}^{n} \langle v, e_j \rangle e_j \right\rangle \right| \\ &= \left| \sum_{i=1}^{n} \lambda_i | \langle v, e_i \rangle |^2 \right| \leq \max_i |\lambda_i| \end{aligned}$$

and the equality happens when v is one of $(e_1, ..., e_n)$. Thus

$$\sup \{ \langle Tv, v \rangle : \|v\| = 1 \} = \max_i |\lambda_i| = \|T\|.$$

□

1.3.4 Convergence of Matrices

Let V be a finite-dimensional inner-product space over $\mathbb{C}$. The set of all operators on V equipped with the operator norm is a metric space and thus we can talk about convergence of operators. Let (B_k) be a sequence of operators on V. If we choose an orthonormal basis $(e_1, ..., e_n)$ for V, then each B_k can be represented as an $n \times n$ complex matrix B_k.

Proposition 1.11. *The sequence* (B_k) *converges to some operator* A *in the operator norm if and only if*

$$\lim_{k\to\infty} (\mathsf{B}_k)_{ij} = \mathsf{A}_{ij}$$

for $1 \le i, j \le n$.

Proof. It suffices to prove the proposition in the case where $A = 0$, i.e.,

$$\lim_{k\to\infty} B_k = 0$$

in the operator norm if and only if

$$\lim_{k\to\infty} (\mathsf{B}_k)_{ij} = 0$$

for $1 \le i, j \le n$. For the forward direction, suppose that

$$\lim_{k\to\infty} B_k = 0$$

in the operator norm, i.e., for every $\varepsilon > 0$, there exists some integer N such that

$$\|B_k\| < \varepsilon \quad \text{for all} \quad k \ge N.$$

Now since $(e_1, ..., e_n)$ is orthonormal, we have

$$|(\mathsf{B}_k)_{ij}| = |\langle B_k e_j, e_i\rangle| \le \|B_k e_j\|\|e_i\| \le \|B_k\| < \varepsilon$$

for $k \ge N$. Therefore

$$\lim_{k\to\infty} (\mathsf{B}_k)_{ij} = 0$$

for $1 \le i, j \le n$.

For the other direction, suppose that

$$\lim_{k\to\infty} (\mathsf{B}_k)_{ij} = 0$$

for $1 \le i, j \le n$. Then for any $\varepsilon > 0$, there exists some integer N such that

$$\sum_{1\le i,j\le n} |(\mathsf{B}_k)_{ij}|^2 < \varepsilon^2 \quad \text{for all} \quad k \ge N.$$

It follows that

$$\|B_k e_j\|^2 = \sum_{i=1}^{n} |(\mathsf{B}_k)_{ij}|^2 < \varepsilon^2 \quad \text{for all} \quad k \ge N$$

for $j = 1, ..., n$. Now for every vector $v \in V$ such that $\|v\| = 1$, we have

$$v = \sum_{j=1}^{n} a_j e_j,$$

where $|a_1|^2 + \cdots + |a_n|^2 = 1$. Hence

$$\|B_k v\|^2 = \langle B_k v, B_k v\rangle = \sum_{j=1}^{n} |a_j|^2 \|B_k e_j\|^2 < \varepsilon^2$$

for $k \geq N$. By the definition of the operator norm, we conclude that

$$\|B_k\| < \varepsilon \quad \text{for all} \quad k \geq N.$$

In other words,

$$\lim_{k\to\infty} B_k = 0.$$

$\square$

Thus if we have

$$A = \sum_{k=0}^{\infty} B_k,$$

then

$$\mathsf{A} = \sum_{k=0}^{\infty} \mathsf{B}_k.$$

(Here the convergence of matrices means the convergence of each entry.) The trace of A doesn't depend on the choice of basis, hence

$$\operatorname{tr} A = \sum_{i=1}^{n} (\mathsf{A})_{ii} = \sum_{i=1}^{n}\sum_{k=0}^{\infty} (\mathsf{B}_k)_{ii} = \sum_{k=0}^{\infty}\sum_{i=1}^{n} (\mathsf{B}_k)_{ii} = \sum_{k=0}^{\infty} \operatorname{tr} B_k.$$

We also note that if B is a self-adjoint operator such that

$$A = \sum_{k=0}^{\infty} a_k B^k,$$

consider an orthonormal basis for V consisting of eigenvectors of B, it follows from Theorem 1.11 that the eigenvalues of A are given by

$$\left\{ \sum_{k=0}^{\infty} a_k \lambda^k : \lambda \text{ is an eigenvalue of } B \right\}.$$

This is in accordance with our definition of $\sum a_k B^k$ given in Subsection 1.2.5.

Proposition 1.12. *The set of all operators on V equipped with the operator norm is a complete metric space, i.e., every Cauchy sequence converges.*

Proof. Consider a Cauchy sequence (B_k), i.e., for every $\varepsilon > 0$, there exists some integer N such that
$$\|B_p - B_q\| < \varepsilon \quad \text{for all} \quad p, q \geq N.$$
Thus for $j = 1, ..., n$,
$$\|(B_p - B_q)e_j\| < \varepsilon \quad \text{for all} \quad p, q \geq N.$$
It follows that
$$|(\mathsf{B}_p)_{ij} - (\mathsf{B}_q)_{ij}| < \varepsilon \quad \text{for all} \quad p, q \geq N$$
for $1 \leq i, j \leq n$. Thus each sequence $(\mathsf{B}_k)_{ij}$ is Cauchy. Since $\mathbb{C}$ is complete, we have
$$\lim_{k\to\infty} (\mathsf{B}_k)_{ij} = \mathsf{A}_{ij}.$$
By Proposition 1.11, we conclude that
$$\lim_{k\to\infty} B_k = A,$$
where A is the operator on V whose matrix representation with respect to $(e_1, ..., e_n)$ is A. □

Proposition 1.13. *If L is an operator on V such that $\|L\| < 1$, then*
$$\frac{1}{1-L} = \sum_{k=0}^{\infty} L^k.$$
Note that $L^0 = I$, the identity operator.

Proof. This proposition is an analogue of a result for complex numbers for the case of operators. It says that if $\|L\| < 1$, then $1 - L$ is invertible and can be represented by a power series of operators. To show the result, we first prove that the series on the right hand side is convergent and then show that its sum is precisely the inverse of $1 - L$. For positive integers $q > p$ we have
$$\left\| \sum_{k=0}^{q} L^k - \sum_{k=0}^{p} L^k \right\| = \left\| \sum_{k=p+1}^{q} L^k \right\| \leq \sum_{k=p+1}^{q} \left\| L^k \right\| \leq \sum_{k=p+1}^{q} \|L\|^k,$$
where the last inequality follows from a property of the operator norm. Because the series
$$\sum_{k=0}^{\infty} \|L\|^k$$
is convergent for $\|L\| < 1$, we conclude that the sequence of partial sums of the series $\sum L^k$ form a Cauchy sequence. By Proposition 1.12, the series converges to some operator S
$$S = \sum_{k=0}^{\infty} L^k.$$

Multiply through both sides with L we obtain

$$LS = \sum_{k=1}^{\infty} L^k = S - I.$$

Thus

$$S = (I - L)^{-1} = \frac{1}{1 - L}.$$

□

1.4 Legendre Polynomials

Consider the inner product space X consisting of all continuous real-valued functions on $[-1, 1]$ with the inner product given by

$$\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt.$$

(Note that X is an infinite-dimensional vector space.) We want to obtain an orthonormal sequence in X which consists of functions that are easy to handle. Polynomials are of this type and we start from the linearly independent sequence of powers:

$$x_0(t) = 1, \quad x_1(t) = t, \quad ..., \quad x_k(t) = t^k, \quad ... \quad t \in [-1, 1].$$

Applying the Gram-Schmidt procedure (Theorem 1.1) we obtain an orthonormal sequence (e_n). Each e_n is clearly a polynomial since it is just a linear combination of (x_n) and we claim that (e_n) have the following explicit form:

$$e_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t) \quad n = 0, 1, ..., \tag{1.1}$$

where

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}[(t^2 - 1)^n].$$

The polynomial P_n is called the *Legendre polynomial of order* n and the formula for P_n given above is known as the *Rodrigues' formula*. The explicit form of the polynomial e_n may be found by applying the Gram-Schmidt procedure directly. However here we present another method. For this the following proposition is crucial.

Proposition 1.14. *Let (y_n) be an orthonormal sequence in X, where each y_n is a polynomial of degree n for $n = 0, 1, 2...$ Then we have*

$$y_n = \pm e_n \quad n = 0, 1, 2...$$

Proof. From the Gram-Schmidt procedure we have

$$Y_n = \operatorname{span}\{x_0, ..., x_n\} = \operatorname{span}\{e_0, ..., e_n\}$$

for all n. Then Y_n is vector space of dimension $n+1$. Since each y_n is a polynomial of degree n, we conclude that

$$\operatorname{span}\{y_0, ..., y_n\} \subset Y_n.$$

Moreover, $y_0, ..., y_n$ are linearly independent since they are orthogonal. Thus

$$Y_n = \operatorname{span}\{y_0, ..., y_n\} \quad n = 0, 1, ...$$

It follows that we can express y_n as a linear combination of e_n:

$$y_n = \sum_{j=0}^{n} \alpha_j e_j, \quad \alpha_j \in \mathbb{R}.$$

Now by the orthogonality

$$y_n \perp Y_{n-1} = \operatorname{span}\{y_0, ..., y_{n-1}\} = \operatorname{span}\{e_0, ..., e_{n-1}\},$$

we obtain for $k = 0, 1, ..., n-1$,

$$0 = \langle y_n, e_k \rangle = \left\langle \sum_{j=0}^{n} \alpha_j e_j, e_k \right\rangle = \alpha_k.$$

Therefore,

$$y_n = \alpha_n e_n$$

for $n = 0, 1, ...$ (The case $n = 0$ can be checked manually.) Since both y_n and e_n are unit vectors, we have

$$1 = \|y_n\| = |\alpha_n| \|e_n\| = |\alpha_n|.$$

Because α_n is real, α_n is ± 1. $\square$

In our case we can let y_n to be the right hand side of (1.1). Then clearly each y_n is a polynomial of degree n since P_n is a polynomial of degree $2n$ and we differentiate it n times. If (y_n) is orthonormal, then

$$y_n = \pm e_n$$

by Proposition 1.14. Furthermore, the coefficient of t^n in e_n is positive (recall the formula of the Gram-Schmidt procedure) and so is the coefficient of t^n in y_n (the coefficient of t^{2n} is positive). By equating the coefficients of t^n in both polynomials we conclude that

$$y_n = e_n$$

for $n = 0, 1, 2...$ Thus now we just need to prove that the sequence (y_n) is orthonormal. For that purpose the following observation is useful.

Proposition 1.15. *Put $u(t) = t^2 - 1$. Then we have*

$$(u^n)^{(k)}(\pm 1) = 0, \quad k = 0, ..., n-1$$

and

$$(u^n)^{(2n)} = (2n)!,$$

where $(u^n)^{(k)}$ denotes the k-th derivative of u^n.

Proof. We recall the general Leibniz rule for repeated differentiation:

$$(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)},$$

which can be proved easily by induction. Applying this formula to u^n we obtain

$$\begin{aligned}
\frac{d^k}{dt^k} u^n &= \frac{d^k}{dt^k}[(t-1)^n (t+1)^n] \\
&= \sum_{j=0}^{k} \binom{k}{j} [(t-1)^n]^{(j)} [(t+1)^n]^{(k-j)} \\
&= \sum_{j=0}^{k} \binom{k}{j} n \cdots (n-j+1)(t-1)^{n-j} n \cdots (n-k+j+1)(t+1)^{n-k+j}.
\end{aligned}$$

Now for $0 \le k \le n-1$, both $n-j$ and $n-(k-j)$ are positive. Hence every term of $(u^n)^{(k)}$ contains the factor $t^2 - 1$ and thus vanishes when $t = \pm 1$. For the second equality, note that

$$(u^n)^{(2n)} = (t^{2n})^{(2n)} = (2n)!.$$

□

Proposition 1.16. *The sequence (y_n) is orthonormal.*

Proof. Put $u(t) = t^2 - 1$. Applying integration by parts n times we obtain

$$\begin{aligned}
(2^n n!)^2 \|P_n\|^2 &= \int_{-1}^{1} (u^n)^{(n)} (u^n)^{(n)} dt \\
&= (u^n)^{(n-1)} (u^n)^{(n)} \Big|_{-1}^{1} - \int_{-1}^{1} (u^n)^{(n+1)} (u^n)^{(n-1)} dt \\
&= -\left((u^n)^{(n+1)} (u^n)^{(n-2)} \Big|_{-1}^{1} - \int_{-1}^{1} (u^n)^{(n+2)} (u^n)^{(n-2)} dt \right) \\
&= \cdots \\
&= (-1)^{n-1} \left((u^n)^{(2n)} (u^n) \Big|_{-1}^{1} - \int_{-1}^{1} (u^n)^{(2n)} u^n dt \right) \\
&= (-1)^n (2n)! \int_{-1}^{1} (t^2 - 1)^n dt \\
&= 2(2n)! \int_{0}^{1} (1 - t^2)^n dt,
\end{aligned}$$

where we've used $(u^n)^{(k)}(\pm 1) = 0$ for $k = 0, ..., n-1$ and $(u^n)^{(2n)} = (2n)!$. Now for the integral

$$I_n = \int_0^1 (1-t^2)^n dt,$$

by integration by parts we obtain the following recurrence relation

$$I_n = \frac{2n}{2n+1} I_{n-1}, \quad n \geq 1.$$

Since $I_0 = 1$ we obtain

$$I_n = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\cdots\frac{2}{3}.$$

Plug in everything we get

$$(2^n n!)^2 \|P_n\|^2 = \frac{2^{2n+1}(n!)^2}{2n+1}.$$

Thus

$$\|P_n\|^2 = \frac{2}{2n+1}.$$

It follows that

$$\|y_n\|^2 = \left\|\sqrt{\frac{2n+1}{2}} P_n(t)\right\|^2 = 1.$$

Now for the orthogonality, we'll show that $\langle P_m, P_n\rangle = 0$ for $0 \leq m < n$. Since P_m is a polynomial, it suffices to prove that $\langle x_m, P_n\rangle = 0$ for $0 \leq m < n$. Applying integration by parts m times we obtain

$$\begin{aligned}
\langle x_m, P_n\rangle &= \int_{-1}^1 t^m (u^n)^{(n)} dt \\
&= t^m (u^n)^{(n-1)}\Big|_{-1}^1 - \int_{-1}^1 m t^{m-1} (u^n)^{(n-1)} dt \\
&= \cdots \\
&= (-1)^m m! \int_{-1}^1 (u^n)^{(n-m)} dt \\
&= (-1)^m m! (u^n)^{(n-m-1)}\Big|_{-1}^1 \\
&= 0,
\end{aligned}$$

where the last equality follows because $0 \leq m < n$. $\square$

We can expand $(t^2-1)^n$ using the binomial theorem:

$$(t^2-1)^n = \sum_{j=0}^n \binom{n}{j} (t^2)^j (-1)^{n-j}.$$

Thus

$$
\begin{aligned}
P_n(t) &= \frac{1}{2^n n!}\frac{d^n}{dt^n}\sum_{j=0}^{n}\binom{n}{j}t^{2j}(-1)^{n-j} \\
&= \frac{1}{2^n n!}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}2j(2j-1)\cdots(2j-n+1)t^{2j-n} \\
&= \frac{1}{2^n n!}\sum_{j=\lceil \frac{n}{2}\rceil}^{n}\binom{n}{j}(-1)^{n-j}2j(2j-1)\cdots(2j-n+1)t^{2j-n}.
\end{aligned}
$$

Here $\lceil x \rceil$ denotes the ceiling of x, i.e., the smallest integer greater than or equal to x. A first few Legendre polynomials are

$$
\left\{1,\, t,\, \frac{1}{2}(3t^2-1),\, \frac{1}{2}(5t^3-3t),\, \frac{1}{8}(35t^4-30t^2+3), ...\right\}.
$$

Proposition 1.17. *The Legendre polynomials satisfy the following recurrence relation:*

$$
\begin{cases} P_0(t) = 1, \\ (n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t), \quad n \geq 1. \end{cases} \tag{1.2}
$$

Proof. The recurrence relation can be checked by direct computation and the proof is quite tedious. We'll show for the case n is odd and the same idea can be applied to when n is even. When n is odd, note that

$$
\left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}, \quad \left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}, \quad \left\lceil \frac{n+1}{2} \right\rceil = \frac{n+1}{2}.
$$

Thus we have the following:

$$
(2n+1)tP_n(t) = \frac{2n+1}{2^n n!}\sum_{j=\frac{n+1}{2}}^{n}\binom{n}{j}(-1)^{n-j}2j(2j-1)\cdots(2j-n+1)t^{2j-n+1},
$$

$$
-nP_{n-1}(t) = \frac{-n}{2^{n-1}(n-1)!}\sum_{j=\frac{n-1}{2}}^{n-1}\binom{n-1}{j}(-1)^{n-j-1}2j(2j-1)\cdots(2j-n+2)t^{2j-n+1},
$$

$$
(n+1)P_{n+1}(t) = \frac{1}{2^{n+1}n!}\sum_{j=\frac{n-1}{2}}^{n}\binom{n+1}{j+1}(-1)^{n-j}(2j+2)(2j+1)\cdots(2j-n+2)t^{2j-n+1}.
$$

Now since the powers of t in the three polynomials are the same, we just need to show that the sum of the first two coefficients gives the third coefficient. (Note that the case $j=(n-1)/2$ and $j=n$ need to be checked separately.) With some patience we should obtain our result. $\square$

Note that $P_0(t) = 1$ for all $t \in [-1, 1]$ and $P_1(1) = 1$. Suppose that $P_{n-1}(1) = P_n(1) = 1$, it follows from the recurrence relation that

$$(n+1)P_{n+1}(1) = (2n+1) - n = n+1.$$

Thus we have $P_n(1) = 1$ for $n = 0, 1, 2, ...$ Moreover, it can be shown that $|P_n(t)| \le 1$ for all n. Hence for a fixed $t \in [-1, 1]$, the following series

$$Q_t(x) = \sum_{n=0}^{\infty} P_n(t) x^n$$

is convergent for all $x \in \mathbb{R}$ and $|x| < 1$. We'd like to find the function $Q_t(x)$. First, differentiate $Q_t(x)$ with respect to x and multiply both sides by x we obtain

$$xQ_t'(x) = \sum_{n=1}^{\infty} nP_n(t)x^n, \quad |x| < 1.$$

Using the recurrence relation (1.2) we obtain

$$\begin{aligned} xQ_t'(x) &= tx + \sum_{n=1}^{\infty}(n+1)P_{n+1}(t)x^{n+1} \\ &= tx + \sum_{n=1}^{\infty}\left[(2n+1)tP_n(t) - nP_{n-1}(t)\right]x^{n+1} \\ &= \sum_{n=1}^{\infty} 2ntP_n(t)x^{n+1} + \sum_{n=1}^{\infty} tP_n(t)x^{n+1} - \sum_{n=1}^{\infty} nP_{n-1}(t)x^{n+1} + tx \\ &= 2tx^2Q_t'(x) + tx(Q_t(x) - 1) - (x^3Q_t'(x) + x^2Q_t(x)) + tx \\ &= (2tx^2 - x^3)Q_t'(x) + (tx - x^2)Q_t(x) \end{aligned}$$

Thus,

$$(1 - 2tx + x^2)Q_t'(x) = (t - x)Q_t(x), \quad |x| < 1.$$

(Note that this holds for $x = 0$ because $Q_t(x)$ is continuous.) Now

$$1 - 2tx + x^2 = (x - t)^2 + (1 - t^2) > 0$$

because $|t| \le 1$ and $|x| < 1$. It follows that $Q_t(x)$ satisfies the initial value problem

$$\begin{cases} Q_t'(x) = \left(\dfrac{t - x}{1 - 2tx + x^2}\right) Q_t(x), \\ Q_t(0) = P_0(t) = 1. \end{cases}$$

By separation of variables we get

$$Q_t(x) = \frac{1}{\sqrt{x^2 - 2tx + 1}}.$$

Therefore,

$$\frac{1}{\sqrt{x^2 - 2tx + 1}} = \sum_{n=0}^{\infty} P_n(t)x^n, \quad |x| < 1, \ |t| \le 1.$$

The function $Q_t(x)$ is called the *generating function* of the Legendre polynomials.

From the generating function perspective, we can derive another important property of the Legendre polynomials. We have

$$\sum_{n=0}^{\infty} P_n(t)(-x)^n = Q_t(-x) = \frac{1}{\sqrt{x^2 + 2tx + 1}} = Q_{-t}(x) = \sum_{n=0}^{\infty} P_n(-t)x^n.$$

By uniqueness of power series expansion we conclude that

$$(-1)^n P_n(t) = P_n(-t).$$

Thus P_n is an odd polynomial if n is odd and is an even polynomial if n is even.

Proposition 1.18. *The Legendre polynomial P_n satisfies the following differential equation:*

$$(x^2 - 1)y'' + 2xy' - n(n+1)y = 0,$$

or equivalently,

$$\left[(x^2 - 1)y'\right]' - n(n+1)y = 0.$$

Proof. Put $u(x) = (x^2 - 1)^n$, then $u^{(n)} = 2^n n! P_n$ and

$$(x^2 - 1)u'(x) = n2x(x^2 - 1)^n = 2nxu(x).$$

Differentiate the above equation $n+1$ times using the general Leibniz rule for repeated differentiation:

$$(fg)^{(k)} = \sum_{j=0}^{k} \binom{n}{k} f^{(k)} g^{(n-k)}$$

we obtain

$$\sum_{j=0}^{n+1} \binom{n+1}{j} (x^2 - 1)^{(j)} (u')^{(n+1-j)} = 2n \sum_{j=0}^{n+1} \binom{n+1}{j} x^{(j)} u^{(n+1-j)}$$

Expand and combine the terms we get our desired differential equation. $\square$

2 A Simple Case of the Atiyah–Singer Index Theorem

2.1 Introduction

The Atiyah–Singer (AS) index theorem is one of the great landmarks of twentieth century mathematics. The theorem was proved by Michael Atiyah and Isadore Singer in 1963, for which they were awarded the Abel prize (the "mathematician's Nobel prize") in 2004. The index theorem includes several important theorems in differential geometry as special cases, such as the Riemann–Roch theorem and has many applications in theoretical physics. In a nutshell, it asserts that we can obtain some information about the number of solutions to a certain type of partial differential equations (a.k.a. the analytical index) by essentially looking at the shape, or topology of the domain (a.k.a. the topological index). In this sense, the index theorem connects two important branches of mathematics: analysis and topology. The general statement of the AS index theorem is quite complicated and lies beyond the scope of this note. Nevertheless, we can get some feeling for the AS index theorem by looking at some elementary examples that illustrate the same idea, i.e., relating analytical information and topological information.

Let V and W be complex vector spaces (think of spaces of smooth functions) and $L\colon V \to W$ be a linear map (think of differential operators). The *cokernel* of L is defined as

$$\operatorname{coker} L = W/\operatorname{Im} L.$$

The map L is called *Fredholm* if both $\ker L$ and $\operatorname{coker} L$ are finite-dimensional. In that case, we define the (analytical) index of L to be

$$\operatorname{Index} L = \dim\ker L - \dim\operatorname{coker} L.$$

At first glance, the index of L is purely analytical since it concerns with the number of (independent) solutions to $Lv = 0$. It's surprising that $\operatorname{Index} L$ also carries topological information. Let's first look at the case where V and W are finite-dimensional. In this case, every linear map L is Fredholm. More specifically, suppose $V = W = \mathbb{C}^2$. We consider three different linear maps L and calculate their indices.

1. $L = 0$. It's clear that $\ker L = \mathbb{C}^2$ and $\operatorname{coker} L = \mathbb{C}^2/\{0\} = \mathbb{C}^2$. Hence $\operatorname{Index} L = 2 - 2 = 0$.

2. $$L = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}.$$

By elementary linear algebra we obtain

$$\ker L = \operatorname{span}\left\{(-3,1)^t\right\}$$

and

$$\operatorname{Im} L = \operatorname{span}\left\{(1,2)^t\right\}.$$

Thus $\operatorname{Index} L = 1 - 1 = 0$.

3. $L = \mathrm{Id}$. Then $\ker L = \{0\}$ and $\operatorname{coker} L = \mathbb{C}^2/\mathbb{C}^2 = \{0\}$. Thus $\operatorname{Index} L = 0 - 0 = 0$.

We see that three completely different linear maps all have the same index! (Their kernels are very different, having different dimensions.) This is in fact an instance of a more general result.

Proposition 2.1. *Let V and W be complex finite-dimensional vector spaces and $L\colon V \to W$ be a linear map. Then*

$$\operatorname{Index} L = \dim V - \dim W.$$

Proof. This is just an application of the famous dimension theorem in linear algebra:

$$\dim V = \dim\ker L + \dim\operatorname{Im} L.$$

Now since $\operatorname{coker} L = W/\operatorname{Im} L$, we have

$$\dim\operatorname{coker} L = \dim W - \dim\operatorname{Im} L.$$

It follows that

$$\operatorname{Index} L = \dim\ker L - \dim\operatorname{coker} L = \dim\ker L - (\dim W - \dim\operatorname{Im} L) = \dim V - \dim W.$$

□

Since dimension of a vector space is a topological property (it is unchanged under homeomorphism), we see that the index of L also encodes topological information. This proposition however doesn't make sense when V and W are infinite-dimensional. In that case L will carry a different type of topological information. Let's look at one example.

Suppose $V = W = C^\infty(\mathbb{S}^1)$, where $C^\infty(\mathbb{S}^1)$ denotes smooth complex-valued functions on $\mathbb{S}^1$, which can be thought of as periodic smooth complex-valued functions on $\mathbb{R}$ with period 2π. Let $L\colon C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ be the differentiation map $f \mapsto f'$. The kernel of L is given by

$$\ker L = \left\{f \in C^\infty(\mathbb{S}^1)\colon f' = 0\right\} = \mathbb{C}$$

by elementary calculus. The calculation of $\operatorname{coker} L$ is more involved. We first show that for $f \in C^\infty(\mathbb{S}^1)$,

$$\int_0^{2\pi} f\,d\theta = 0 \Leftrightarrow f \in \operatorname{Im} L.$$

If $f \in \operatorname{Im} L$, then $f = F'$ for some periodic function F with period 2π. It follows that

$$\int_0^{2\pi} f d\theta = \int_0^{2\pi} F' d\theta = F(2\pi) - F(0) = 0.$$

Conversely, if

$$\int_0^{2\pi} f d\theta = 0,$$

we want to find a function periodic function F with period 2π such that $F' = f$. Inspired by the fundamental theorem of calculus, we can put F to be

$$F(x) = \int_0^x f d\theta.$$

Then clearly $F' = f$. To see that F is periodic, note that

$$F(x+2\pi) - F(x) = \int_x^{x+2\pi} f d\theta = \int_x^0 f d\theta + \int_0^{2\pi} f d\theta + \int_{2\pi}^{x+2\pi} f d\theta = \int_0^{2\pi} f d\theta = 0,$$

where the third equality follows because f is periodic. Let $[f]$ denote the equivalence class of f in $C^\infty(\mathbb{S}^1)/\operatorname{Im} L$. Thus $[f] = [g]$ means $f - g \in \operatorname{Im} L$. Consider the following linear map $T\colon C^\infty(\mathbb{S}^1)/\operatorname{Im} L \to \mathbb{C}$ given by

$$[f] \mapsto \int_0^{2\pi} f d\theta.$$

Note that this map is well-defined since if $[f] = [g]$, then

$$\int_0^{2\pi} (f-g) d\theta = \int_0^{2\pi} h d\theta = 0$$

since $h \in \operatorname{Im} L$. The map T is clearly nonzero. To show that T is an isomorphism, it suffices to show that T is one-to-one. If follows because

$$\int_0^{2\pi} (f-g) d\theta = 0 \Rightarrow f - g \in \operatorname{Im} L.$$

Thus $[f] = [g]$ and so $\operatorname{coker} T \cong \mathbb{C}$. Hence the index of L is $1 - 1 = 0$.

What topological information does the index of L carry in this case? Recall the *Euler characteristic* of the circle $\mathbb{S}^1$, which is a topological invariant. To calculate the Euler characteristic of $\mathbb{S}^1$, put a finite number of dots on the circle and count the number of dots subtract the number of edges formed between adjacent dots. The result doesn't depend on how many dots are put on the circle. For instance, if we put 3 dots on the circle, then we have 3 edges and the Euler characteristic is $3 - 3 = 0$, which is precisely the index of L!

We give one final example. Consider the following operator

$$L_{a,b} = a\frac{d}{dx} + b\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),$$

where a and b are arbitrary complex numbers and $a \neq 0$. The kernel of $L_{a,b}$ consists of the solutions to the differential equation

$$a\frac{d}{dx}f + bf = 0.$$

By separation of variables we obtain

$$f(x) = Ce^{-\frac{b}{a}x}, \quad C \in \mathbb{C}.$$

Thus $\dim\ker L_{a,b} = 1$. To find the cokernel of $L_{a,b}$, we observe that for any $f \in C^\infty(\mathbb{R})$, if we put

$$F(x) = e^{-\frac{b}{a}x}\left(\frac{1}{a}\int_0^x e^{\frac{b}{a}t} f(t)dt\right),$$

then

$$a\frac{d}{dx}F + bF = f.$$

Thus $\operatorname{Im} L_{a,b} = C^\infty(\mathbb{R})$. It follows that $\dim\operatorname{coker} L_{a,b} = 0$ and hence $\operatorname{Index} L_{a,b} = 1 - 0 = 1$. This example illustrates the fact that the index is stable under deformations. Moreover, the number 1 is also the Euler characteristic of the real line $\mathbb{R}$. (To compute the Euler characteristic of $\mathbb{R}$, we proceed analogously as in the case of $\mathbb{S}^1$. However, we need to discard the two unbounded edges.)

The AS index theorem also finds its applications in Quantum Chromodynamics (QCD), which is a physics theory that studies the behaviors of various subnuclear particles interacting via the strong nuclear force, one of the four fundamental forces of nature. The AS index theorem plays an important role in explaining why certain particles have a larger mass than expected. In the remaining sections, we aim to describe a simple case of the AS index theorem within the context of gauge theory in theoretical physics. We first need the concept of gauge fields (a.k.a. connections in the mathematics literature).

2.2 Gauge Fields

Consider the complex vector space consisting of all smooth functions $\mathbb{R}^2 \to \mathbb{C}^2$. Recall that a function $\mathbb{R}^2 \to \mathbb{C}$ is called smooth if its real and imaginary parts are smooth in the usual sense, i.e., partial derivatives of all orders exist and are continuous. A function $\mathbb{R}^2 \to \mathbb{C}^2$ is smooth if each component function is smooth. Let V denote the vector subspace consisting of all smooth functions $\mathbb{R}^2 \to \mathbb{C}^2$ satisfying the following *twisted periodicity conditions*:

$$\begin{cases} f(x_1+1, x_2) = f(x_1, x_2), \\ f(x_1, x_2+1) = g(x_1)f(x_1, x_2), \end{cases} \tag{2.1}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$ and $g(x_1) \in \mathrm{U}(1)$. Here $\mathrm{U}(1)$ is the multiplicative group of complex numbers whose moduli are 1:

$$\mathrm{U}(1) = \{z \in \mathbb{C}\colon |z| = 1\}.$$

Topologically, $\mathrm{U}(1)$ is the same as $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$. Since g is continuous, there exists a continuous function $\theta\colon \mathbb{R} \to \mathbb{R}$ such that

$$g(x_1) = e^{i\theta(x_1)}.$$

From condition (2.1) we deduce that

$$\begin{aligned} f(x_1+1, x_2+1) &= f(x_1, x_2+1) = e^{i\theta(x_1)} f(x_1, x_2) \\ &= e^{i\theta(x_1+1)} f(x_1+1, x_2) = e^{i\theta(x_1+1)} f(x_1, x_2). \end{aligned}$$

Hence

$$\theta(x_1+1) = \theta(x_1) + 2\pi Q$$

for all $x_1 \in \mathbb{R}$ and Q is some integer. (To be more precise, Q should depend on x_1. However, since θ is continuous and Q is an integer, it is constant.) With that condition the map $\mathbb{R} \to \mathbb{C}$ given by $x_1 \mapsto e^{i\theta(x_1)}$ is just a continuous map $\mathbb{S}^1 \to \mathbb{S}^1$ and the integer Q is just the winding number (the number of times the map wraps around $\mathbb{S}^1$), which characterizes homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$. For an example of V, we can choose $\theta(x_1) = i2\pi Q x_1$ for some integer Q. An element of V is

$$f(x_1, x_2) = \begin{pmatrix} e^{i2\pi x_1(1+Qx_2)} \\ e^{i2\pi x_1(2+Qx_2)} \end{pmatrix}.$$

It's clear that f satisfies the twisted periodicity condition. (To find such a function, write $f_j(x_1, x_2) = e^{i(p_1x_1+p_2x_2)}$ and use the twisted condition to specify p_1 and p_2.) Finally, we can turn V into an inner product space by

$$\langle f, g\rangle = \int_{[0,1]^2} g(x)^* f(x) dx \ \text{ for all } f, g \in V,$$

where $g(x)^*$ denotes the conjugate transpose of $g(x)$ (recall that $g(x)$ is a two-component column vector) .

We want to define a differential operator on V. However, as we can see, the partial derivatives don't preserve the condition (2.1):

$$\partial_1 f(x_1, x_2+1) = \partial_1(e^{i\theta(x_1)} f(x_1, x_2)) = e^{i\theta(x_1)}(i\theta'(x_1) f(x_1, x_2) + \partial_1 f(x_1, x_2)),$$

which is different from $e^{i\theta(x_1)}\partial_1(x_1, x_2)$. To remedy the situation, we introduce the concept of *gauge fields*. A gauge field A can be thought of as a collection of smooth functions $A_\mu\colon \mathbb{R}^2 \to \mathbb{R}$ for $\mu = 1, 2$. We replace the partial derivatives by

$$\nabla_\mu \colon = \partial_\mu + iA_\mu \ \text{ for } \mu = 1, 2.$$

We want the ∇_μ to map V to itself, i.e.,

$$\nabla_\mu f(x_1+1, x_2) = \nabla_\mu f(x_1, x_2), \tag{2.2}$$

$$\nabla_\mu f(x_1, x_2+1) = e^{i\theta(x_1)} \nabla_\mu f(x_1, x_2) \tag{2.3}$$

for all $f \in V$ and $\mu = 1, 2$. These requirements will impose certain conditions on the gauge field, as we are about to find out. Let's first consider requirement (2.2), writing everything out we obtain

$$\partial_\mu f(x_1 + 1, x_2) + iA_\mu(x_1 + 1, x_2) f(x_1 + 1, x_2) = \partial_\mu f(x_1, x_2) + iA_\mu(x_1, x_2) f(x_1, x_2).$$

Plugging in $f(x_1 + 1, x_2) = f(x_1, x_2)$ yields

$$A_\mu(x_1 + 1, x_2) = A_\mu(x_1, x_2) \text{ for } \mu = 1, 2.$$

Now for requirement (2.3), writing everything out we obtain

$$\begin{aligned} &\partial_\mu f(x_1, x_2 + 1) + iA_\mu(x_1, x_2 + 1) f(x_1, x_2 + 1) \\ &\qquad = e^{i\theta(x_1)}(\partial_\mu f(x_1, x_2) + iA_\mu(x_1, x_2) f(x_1, x_2)). \end{aligned}$$

For this case we need to consider $\mu = 1$ and $\mu = 2$. For $\mu = 1$, plugging in $f(x_1, x_2+1) = e^{i\theta(x_1)} f(x_1, x_2)$ yields

$$\begin{aligned} &\partial_1(e^{i\theta(x_1)} f(x_1, x_2)) + iA_1(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2) \\ &\qquad = e^{i\theta(x_1)}(\partial_1 f(x_1, x_2) + iA_1(x_1, x_2) f(x_1, x_2)). \end{aligned}$$

The left hand side of the above equation becomes

$$e^{i\theta(x_1)}(i\theta'(x_1) f(x_1, x_2) + \partial_1 f(x_1, x_2)) + iA_1(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2).$$

Equating with the right hand side we get

$$A_1(x_1, x_2 + 1) = A_1(x_1, x_2) - \theta'(x_1).$$

For the case $\mu = 2$ we have

$$\begin{aligned} &\partial_2(e^{i\theta(x_1)} f(x_1, x_2)) + iA_2(x_1, x_2 + 1)e^{i\theta(x_1)} f(x_1, x_2) \\ &\qquad = e^{i\theta(x_1)}(\partial_2 f(x_1, x_2) + iA_2(x_1, x_2) f(x_1, x_2)). \end{aligned}$$

Expanding everything we obtain

$$A_2(x_1, x_2 + 1) = A_2(x_1, x_2).$$

To summarize, the gauge field A needs to satisfy

$$\begin{cases} A_1(x_1 + 1, x_2) = A_1(x_1, x_2), \\ A_1(x_1, x_2 + 1) = A_1(x_1, x_2) - \theta'(x_1), \\ A_2(x_1 + 1, x_2) = A_2(x_1, x_2), \\ A_2(x_1, x_2 + 1) = A_2(x_1, x_2). \end{cases} \tag{2.4}$$

For an example, if we take $\theta(x_1) = 2\pi Q x_1$ for some integer Q. It can be verified that the following smooth functions

$$\begin{cases} A_1(x_1, x_2) = -2\pi Q x_2, \\ A_2(x_1, x_2) = 0 \end{cases}$$

satisfies condition (2.4) of a smooth gauge field. The choice of a gauge field is however highly non-unique. For instance, we can choose

$$\begin{cases} \widehat{A}_1(x_1, x_2) = -2\pi Q x_2 + \sin(2\pi(x_1 + x_2)), \\ \widehat{A}_2(x_1, x_2) = \sin(2\pi(x_1 + x_2)). \end{cases}$$

Then we have

$$\widehat{A}_1(x_1+1, x_2) = -2\pi Q x_2 + \sin(2\pi(x_1+1+x_2)) = -2\pi Q x_2 + \sin(2\pi(x_1+x_2)) = \widehat{A}_1(x_1, x_2),$$

and

$$\begin{aligned} \widehat{A}_1(x_1, x_2+1) &= -2\pi Q(x_2+1) + \sin(2\pi(x_1 + x_2 + 1)) \\ &= -2\pi Q x_2 + \sin(2\pi(x_1 + x_2)) - 2\pi Q \\ &= \widehat{A}_1(x_1, x_2) - \theta'(x_1). \end{aligned}$$

Clearly $\widehat{A}_2(x_1, x_2)$ is periodic in both directions. In general, we can choose $\widehat{A}_\mu$ to be

$$\widehat{A}_\mu(x_1, x_2) = A_\mu(x_1, x_2) + \varphi(x_1, x_2),$$

where $\varphi(x_1, x_2)$ is periodic in x_1 and x_2.

2.3 Gauge Transformations

One important type of operations associated with gauge fields is known as gauge transformations. A *gauge transformation* is a smooth map $\mathbb{R}^2 \to \mathrm{U}(1)$. We can write this map as $e^{i\phi(x)}$ for some smooth function ϕ. Our vector space will be transformed into

$$\widetilde{V} = \left\{ e^{i\phi} f \colon f \in V \right\}.$$

(Note however that elements of $\widetilde{V}$ no longer satisfy the twisted periodicity condition.) The gauge field A will also be transformed into $\widetilde{A}$ according to:

$$\widetilde{\nabla}_\mu(e^{i\phi} f) = e^{i\phi} \nabla_\mu f, \quad f \in V,$$

where $\widetilde{\nabla}_\mu = \partial_\mu + i\widetilde{A}_\mu$. Expanding the left hand side we obtain

$$\begin{aligned} \widetilde{\nabla}_\mu(e^{i\phi} f) &= \partial_\mu(e^{i\phi} f) + i e^{i\phi} \widetilde{A}_\mu f \\ &= i(\partial_\mu \phi)(e^{i\phi} f) + e^{i\phi} \partial_\mu f + i e^{i\phi} \widetilde{A}_\mu f \\ &= e^{i\phi}[\partial_\mu + i(\widetilde{A}_\mu + \partial_\mu \phi)] f. \end{aligned}$$

Thus $\widetilde{A}_\mu = A_\mu - \partial_\mu \phi$.

2.4 Topological Charge

Although we have many choices for a gauge field, they all share the following important property.

Proposition 2.2. *The following quantity*

$$\frac{1}{2\pi}\int_{[0,1]^2} F_{12}(x_1,x_2)dx_1dx_2,$$

where

$$F_{12}=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}$$

is an integer and doesn't depend on the choice of gauge fields.

Proof. We perform a direct computation:

$$\begin{aligned}\int_{[0,1]^2}\left(\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}\right)dx_1dx_2 &= \int_{[0,1]^2}\frac{\partial A_2}{\partial x_1}dx_1dx_2-\int_{[0,1]^2}\frac{\partial A_1}{\partial x_2}dx_2dx_1\\ &= \int_0^1\left(A_2(1,x_2)-A_2(0,x_2)\right)dx_2\\ &\qquad -\int_0^1\left(A_1(x_1,1)-A_1(x_1,0)\right)dx_1\\ &= \int_0^1\theta'(x_1)dx_1=\theta(1)-\theta(0)\\ &= 2\pi Q,\end{aligned}$$

where we've used the twisted periodicity conditions (2.4) in the third equality. The integer Q obtained is precisely the winding number of $x_1\mapsto e^{i\theta(x_1)}$ and hence doesn't depend on the gauge field. □

We call the integer Q the *topological charge* of the gauge field A. It's worthwhile to see how Q changes under a gauge transformation. Note that

$$\begin{aligned}\partial_1\widetilde{A}_2-\partial_2\widetilde{A}_1 &= (\partial_1A_2-\partial_1\partial_2\phi)-(\partial_2A_1-\partial_2\partial_1\phi)\\ &= \partial_1A_2-\partial_2A_1\\ &= F_{12},\end{aligned}$$

where we've used $(\partial_1\partial_2-\partial_2\partial_1)\phi=0$ since ϕ is smooth. Thus Q is unchanged under a gauge transformation. We say that Q is *gauge invariant*.

2.5 Dirac Operators

In this section we describe a particular class of differential operators which are going to be our main focus, namely the *Dirac operators*. In order to define a Dirac operator, we first need the notion of the principal symbol of a differential operator. Roughly speaking, the *principal symbol* of a differential operator is a matrix of polynomials determined by the most "important part" of the operator. More precisely, let L be a differential operator of order m on smooth functions $\mathbb{R}^n \to \mathbb{C}$ (the highest partial derivative is of order m). The *principal symbol* of L is a matrix of n real variables $\xi_1, ..., \xi_n$ obtained by replacing in the highest ordered terms of L:

$$\partial_{x_1} \text{ by } i\xi_1,\ \partial_{x_2} \text{ by } i\xi_2,\ ...,\ \partial_{x_n} \text{ by } i\xi_n.$$

The resulting principal symbol is denoted by

$$\sigma(L)(\xi_1, ..., \xi_n) \ \text{ or } \ \sigma(L)(\xi),$$

where $\xi = (\xi_1, ..., \xi_n)$. Let's calculate the principal symbols of some operators.

1. Consider the operator
$$L_1 = -\partial_x^2 - \partial_y + \sin(x^2 + y^2). \tag{2.5}$$
According to our definition, its principal symbol is given by
$$\sigma(L_1)(\xi) = -(i\xi_1)^2 = \xi_1^2.$$
Note that we only consider the highest ordered terms.

2. Consider the Laplacian
$$\Delta = -\partial_x^2 - \partial_y^2. \tag{2.6}$$
Its principal symbol is
$$\sigma(\Delta)(\xi) = -(i\xi_1)^2 - (i\xi_2)^2 = \|\xi\|^2.$$

3. Consider the Cauchy-Riemann operator
$$D_{CR} = \partial_x + i\partial_y. \tag{2.7}$$
Its principal symbol is
$$\sigma(D_{CR})(\xi) = i\xi_1 + i(i\xi_2) = i\xi_1 - \xi_2.$$

4. Finally, consider the following operator L_{GB} acting on pairs of smooth functions $\mathbb{R}^2 \to \mathbb{C}$:
$$L_{GB}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \partial_y f - \partial_x g \\ \partial_x f + \partial_y g \end{pmatrix}. \tag{2.8}$$
We can rewrite L_{GB} in matrix form
$$L_{GB} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\partial_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_y.$$
Hence the principal symbol of L_{GB} is given by
$$\sigma(L_{GB})(\xi) = \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix}.$$

A differential operator L is called *elliptic* if its principal symbol $\sigma(L)(\xi)$ is invertible for all $\xi \neq 0$. For instance, from the examples above, the Laplacian and the Cauchy-Riemann operator are clearly elliptic. The operator L_1 is not elliptic since $\sigma(L_1)(0,1) = 0$ is not invertible. For the operator L_{GB}, we can compute the determinant of its principal symbol:

$$\det(\sigma(L_{GB})(\xi)) = -\xi_2^2 - \xi_1^2 = -\|\xi\|^2 \neq 0$$

for $\xi \neq 0$. Thus L_{GB} is elliptic. Now we are ready to give our main definition. A first order differential operator L is called a *Dirac operator* if the principal symbol of L satisfies

$$\sigma(L)(\xi)^* \sigma(L)(\xi) = \|\xi\|^2.$$

Note that for $\xi \neq 0$ we can rewrite the above equation as

$$\frac{1}{\|\xi\|^2} \sigma(L)(\xi)^* \sigma(L)(\xi) = 1.$$

Therefore $\sigma(L)(\xi)$ is invertible for $\xi \neq 0$ and a Dirac operator is elliptic. Also, a Dirac operator is first order, so for instance L_1 and the Laplacian are not Dirac operators. Let's see whether the Cauchy-Riemann operator satisfies the requirement of a Dirac operator, we have

$$\sigma(D_{CR})(\xi)^* \sigma(D_{CR})(\xi) = |i\xi_1 - \xi_2|^2 = \|\xi\|^2.$$

Thus D_{CR} is a Dirac operator. Finally, for the operator L_{GB}:

$$\sigma(L_{GB})(\xi)^* \sigma(L_{GB})(\xi) = \begin{pmatrix} -i\xi_2 & -i\xi_1 \\ i\xi_1 & -i\xi_2 \end{pmatrix} \begin{pmatrix} i\xi_2 & -i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} = \|\xi\|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence L_{GB} is also a Dirac operator.

The Dirac operator of our interest is

$$D = \gamma_1 \nabla_1 + \gamma_2 \nabla_2,$$

where

$$\nabla_\mu = \partial_\mu + iA_\mu \ \text{ for } \mu = 1, 2.$$

Here γ_μ are any 2×2 complex matrices that satisfy

$$\begin{cases} \gamma_\mu^* = \gamma_\mu, & \mu = 1, 2, \\ \gamma_\mu^2 = 1, & \mu = 1, 2, \\ \gamma_1 \gamma_2 = -\gamma_2 \gamma_1. \end{cases} \tag{2.9}$$

For instance, we can choose γ_μ to be

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Another choice of γ_μ is given by

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}.$$

Note that the gamma matrices can also be considered as operators on the space of smooth functions V in the usual way:

$$(\gamma_\mu f)(x) = \gamma_\mu f(x), \quad \text{for } x \in \mathbb{R}^2$$

and $f \in V$. The right hand side is just matrix multiplication. (It's usually clear from the context whether we refer to γ_μ as a matrix or as an operator.) From the relation $\gamma_\mu^2 = 1$ It follows immediately that γ_μ as an operator on V is also invertible, with the inverse given by

$$(\gamma_\mu^{-1} f)(x) = \gamma_\mu^{-1} f(x) \quad \text{for } x \in \mathbb{R}^2$$

and $f \in V$. It should be noted that different choices of γ_μ give us different operators. However, they all satisfy the requirement of a Dirac operator:

$$\sigma(D)(\xi)^* \sigma(D)(\xi) = (-\gamma_1^* i\xi_1 - \gamma_2^* i\xi_2)(\gamma_1 i\xi_1 + \gamma_2 i\xi_2) = \|\xi\|^2,$$

where we've applied properties (2.9).

2.6 Chirality

Our main goal of this section is to define the *(analytical) index* of Dirac operators. For our discussion we need the following theorem, which we're going to state without proof.

Theorem 2.1. *Let D be the Dirac operator (as defined in section 2.5) acting on the space of smooth functions satisfying the twisted periodicity condition* (2.1)*. Then* $\ker D$ *is finite dimensional.*

The following matrix

$$\gamma_3 = -i\gamma_1\gamma_2$$

is called a *chirality matrix* in the physics literature. As a matrix, γ_3 satisfy

$$\begin{cases} \gamma_3^* = \gamma_3, \\ \gamma_3^2 = 1, \\ \gamma_3\gamma_\mu = -\gamma_\mu\gamma_3, \quad \mu = 1, 2, \\ \operatorname{tr}(\gamma_3) = 0. \end{cases}$$

These properties follow directly from definition of γ_3 and (2.9). For instance,

$$\gamma_3^* = (-i\gamma_1\gamma_2)^* = i\gamma_2^*\gamma_1^* = i\gamma_2\gamma_1 = -i\gamma_1\gamma_2 = \gamma_3.$$

For the second property, we have

$$\gamma_3^2 = (-i\gamma_1\gamma_2)^2 = -\gamma_1\gamma_2\gamma_1\gamma_2 = \gamma_1^2\gamma_2^2 = 1.$$

For the third property, for $\mu = 1$,

$$\gamma_3\gamma_1 = -i\gamma_1\gamma_2\gamma_1 = i\gamma_1\gamma_1\gamma_2 = -\gamma_1\gamma_3,$$

and for $\mu = 2$,

$$\gamma_3\gamma_2 = -i\gamma_1\gamma_2\gamma_2 = i\gamma_2\gamma_1\gamma_2 = -i\gamma_2\gamma_3.$$

Finally,

$$\mathrm{tr}\,(\gamma_3) = \mathrm{tr}\,(-i\gamma_1\gamma_2) = \mathrm{tr}\,(i\gamma_2\gamma_1) = \mathrm{tr}\,(i\gamma_1\gamma_2) = -\mathrm{tr}\,(\gamma_3).$$

Thus $\mathrm{tr}\,(\gamma_3) = 0$. One important property of γ_3 is the following:

$$\gamma_3 D = -D\gamma_3.$$

This can be seen by direct calculation. We have

$$\begin{aligned}\gamma_3 D &= \gamma_3(\gamma_1\nabla_1 + \gamma_2\nabla_2) = \gamma_3\gamma_1\nabla_1 + \gamma_3\gamma_2\nabla_2\\ &= -\gamma_1\gamma_3\nabla_1 - \gamma_2\gamma_3\nabla_2 = -(\gamma_1\nabla_1 + \gamma_2\nabla_2)\gamma_3\\ &= -D\gamma_3.\end{aligned}$$

(Note that γ_μ as operators on V commute with ∇_ν because ∇_ν act on the elements of V component-wise.) One important consequence of the above property is that $\ker D$ (known to be finite dimensional) is invariant under γ_3, i.e.,

$$\gamma_3\psi \in \ker D, \quad \text{for all } \psi \in \ker D.$$

To see why it's true, take any $\psi \in \ker D$, then

$$D(\gamma_3\psi) = -\gamma_3 D(\psi) = 0.$$

Thus $\gamma_3\psi \in \ker D$. Now we define the projection operators

$$P_+ = \frac{1}{2}(I + \gamma_3), \qquad P_- = \frac{1}{2}(I - \gamma_3).$$

We should check that $P^\pm$ are indeed projections:

$$P_+^2 = \frac{1}{4}(I + \gamma_3)^2 = \frac{1}{4}(I + 2\gamma_3 + \gamma_3^2) = \frac{1}{2}(I + \gamma_3) = P_+,$$

and

$$P_-^2 = \frac{1}{4}(I - \gamma_3)^2 = \frac{1}{4}(I - 2\gamma_3 + \gamma_3^2) = \frac{1}{2}(I - \gamma_3) = P_-,$$

where we've used $\gamma_3^2 = 1$. Other properties of $P_\pm$ are

$$P_+ + P_- = \frac{1}{2}(I + \gamma_3) + \frac{1}{2}(I - \gamma_3) = I,$$

and

$$P_+P_- = \frac{1}{4}(I+\gamma_3)(I-\gamma_3) = 0 = P_-P_+.$$

Put $(\ker D)_\pm = P_\pm(\ker D)$. Since $\ker D$ is invariant under γ_3, we know that $(\ker D)_\pm$ are subspaces of $\ker D$. Even more is true:

Proposition 2.3.

$$\ker D = (\ker D)_+ \oplus (\ker D)_-.$$

Proof. Take any $\psi \in \ker D$, we have

$$\psi = I(\psi) = (P_+ + P_-)\psi = P_+\psi + P_-\psi.$$

To show the intersection of $(\ker D)_+$ and $(\ker D)_-$ is trivial, take any ψ in the intersection, then

$$\psi = P_+\phi_1 = P_-\phi_2,$$

for some $\phi_1, \phi_2 \in \ker D$. Now apply the properties of $P_\pm$ we obtain

$$\psi = P_+\phi_1 = P_+^2\phi_1 = P_+P_-\phi_2 = 0.$$

□

Note that for an element $\psi_+ \in (\ker D)_+$,

$$\gamma_3\psi_+ = \gamma_3 P_+\phi = \gamma_3\frac{1}{2}(I+\gamma_3)\phi = \frac{1}{2}(I+\gamma_3)\phi = \psi_+,$$

and for an element $\psi_- \in (\ker D)_-$,

$$\gamma_3\psi_- = \gamma_3 P_-\varphi = \gamma_3\frac{1}{2}(I-\gamma_3)\varphi = \frac{1}{2}(\gamma_3 - I)\varphi = -\psi_-.$$

For this reason, an element in $(\ker D)_+$ is said to have *positive chirality* and an element in $(\ker D)_-$ is said to have *negative chirality* (corresponding to the eigenvalues 1 and -1 of γ_3, respectively). Now the *(analytical) index* of D is defined to be the quantity

$$\mathrm{Index}\,(D) = \dim(\ker D)_+ - \dim(\ker D)_-.$$

It follows that $\mathrm{Index}\,(D)$ can also be expressed by

$$\mathrm{Index}\,(D) = \mathrm{tr}\,(\gamma_3\colon\ \ker D \to \ker D).$$

Under a gauge transformation $e^{i\phi}$, we have

$$\widetilde{D}(e^{i\phi}f) = \sum_\mu \gamma_\mu \widetilde{\nabla}_\mu(e^{i\phi}f) = e^{i\phi}\sum_\mu \gamma_\mu\nabla_\mu f = e^{i\phi}Df,$$

where the second equality follows from a property of gauge transformations. It follows that

$$\widetilde{D}(e^{i\phi}f) = 0 \Leftrightarrow Df = 0.$$

Thus

$$\ker \widetilde{D} = e^{i\phi} \ker D.$$

Consequently,

$$(\ker \widetilde{D})_+ = e^{i\phi}(\ker D)_+, \quad (\ker \widetilde{D})_- = e^{i\phi}(\ker D)_-.$$

We conclude that $\mathrm{Index}\,\widetilde{D}$ is the same as $\mathrm{Index}\,D$ and hence $\mathrm{Index}\,D$ is gauge invariant.

2.7 The Atiyah–Singer Index Theorem

Regarding the index of the Dirac operator D, we have the following surprising fact:

Theorem 2.2.

$$\mathrm{Index}\,(D) = -Q.$$

Recall that Q is the topological charge (section 2.2) associated with our data. Thus the index of D does not depend on our choice of the gamma matrices and gauge fields! (The dimensions of $(\ker D)_\pm$ may be dependent on these factors but their difference is always a constant.) This theorem is amazing since it tells us that we can extract some information about the number of solutions of the system essentially by looking at the shape, or topology, of the domain, without having to actually find the solutions. In particular, if $Q \neq 0$, then we know that the system has to have some solutions. This is a special case of the celebrated Atiyah-Singer Index Theorem. The general proof is complicated and uses a different technique. In what follows, we attempt to illustrate the theorem for the case of no gauge field $A = 0$, the topological charge $Q = 0$ and the functions are periodic, i.e., $f(x + e_\mu) = f(x)$ for $\mu = 1, 2$.

We first need to talk about hermitian (self-adjoint) operators in infinite-dimensional vector spaces. Similar to the finite-dimensional case, an operator T on V, the space of periodic functions (infinite-dimensional), is called *hermitian* if

$$\langle Tf, g\rangle = \langle f, Tg\rangle \quad \text{for all } f, g \in V.$$

Analogously, an operator T is *anti-hermitian* if

$$\langle Tf, g\rangle = -\langle f, Tg\rangle \quad \text{for all } f, g \in V.$$

In other words,

$$T^* = T.$$

(We will not discuss the notion of adjoint in general vector space in this note, which requires the *Riesz representation theorem*. More details can be found in [14].) We first show that the gamma matrices considered as operators on V are hermitian. Indeed, by properties of the gamma matrices:

$$\begin{aligned}\langle \gamma_\mu f, g\rangle &= \int_{[0,1]^2} g(x)^* \gamma_\mu f(x) dx = \int_{[0,1]^2} g(x)^* \gamma_\mu^* f(x) dx \\ &= \int_{[0,1]^2} (\gamma_\mu g(x))^* f(x) dx = \langle f, \gamma_\mu g\rangle .\end{aligned}$$

Slightly more involved is the following proposition, which is true when we restrict ourselves to functions satisfying the twisted periodicity condition.

Proposition 2.4. *The operators ∂_μ are anti-hermitian.*

Proof. Since the functions are periodic in one direction and twisted periodic in the other direction, we need to consider $\mu = 1$ and $\mu = 2$ separately. First for $\mu = 1$ we have

$$\begin{aligned}
\langle \partial_1 f, g \rangle &= \int_{[0,1]^2} g(x)^* \partial_1 f(x) dx \\
&= \int_{[0,1]^2} \left[\partial_1 (g(x)^* f(x)) - (\partial_1 g(x))^* f(x) \right] dx \quad \text{(by Leibniz rule)} \\
&= \int_0^1 g(x)^* f(x) \Big|_{x_1=0}^{x_1=1} dx_2 - \int_{[0,1]^2} (\partial_1 g(x))^* f(x) dx \\
&= - \int_{[0,1]^2} (\partial_1 g(x))^* f(x) dx \\
&= - \langle f, \partial_1 g \rangle ,
\end{aligned}$$

where we have used $g(1,x_2)^* f(1,x_2) = g(0,x_2)^* f(0,x_2)$. Similarly, for $\mu = 2$,

$$\begin{aligned}
\langle \partial_2 f, g \rangle &= \int_{[0,1]^2} g(x)^* \partial_2 f(x) dx \\
&= \int_{[0,1]^2} \left[\partial_2 (g(x)^* f(x)) - (\partial_2 g(x))^* f(x) \right] dx \quad \text{(by Leibniz rule)} \\
&= \int_0^1 g(x)^* f(x) \Big|_{x_2=0}^{x_2=1} dx_1 - \int_{[0,1]^2} (\partial_2 g(x))^* f(x) dx \\
&= - \int_{[0,1]^2} (\partial_2 g(x))^* f(x) dx \\
&= - \langle f, \partial_2 g \rangle .
\end{aligned}$$

Note that here we have

$$\begin{aligned}
&g(x_1,1)^* f(x_1,1) - g(x_1,0)^* f(x_1,0) \\
&\qquad = e^{-i\theta(x_1)} g(x_1,0)^* e^{i\theta(x_1)} f(x_1,0) - g(x_1,0)^* f(x_1,0) = 0.
\end{aligned}$$

□

One important consequence of the above result is

Corollary 2.1. *The Dirac operator is anti-hermitian, i.e.,*

$$D^* = -D.$$

Proof. The Dirac operator is given by

$$D = \gamma_1 \nabla_1 + \gamma_2 \nabla_2.$$

Now

$$\nabla_\mu^* = (\partial_\mu + iA_\mu)^* = -\partial_\mu - iA_\mu = -\nabla_\mu.$$

Thus

$$D^* = -\gamma_1 \nabla_1 - \gamma_2 \nabla_2 = -D,$$

where we've used the fact that γ_μ are hermitian. □

Now let's return to our case with $A = 0$, $Q = 0$. We want to show that

$$\text{Index}\,(D) = 0.$$

We need to investigate the kernel of D. The main theorem is

Theorem 2.3.

$$D\psi = 0 \Leftrightarrow D^* D\psi = 0 \Leftrightarrow \nabla_\mu^* \nabla_\mu \psi = 0 \Leftrightarrow \nabla_\mu \psi = 0.$$

Proof. We first prove the first equivalence. Clearly $D\psi = 0$ implies $D^* D\psi = 0$. For the other direction, assume that $D^* D\psi = 0$, then

$$\|D\psi\|^2 = \langle D\psi, D\psi \rangle = \langle \psi, D^* D\psi \rangle = \langle \psi, 0 \rangle = 0.$$

Thus $D\psi = 0$.

For the second equivalence, note that

$$\begin{aligned}
D^* D = -D^2 &= -\left(\gamma_1 \nabla_1 + \gamma_2 \nabla_2\right)^2 \\
&= -\left(\gamma_1^2 \nabla_1^2 + \gamma_1 \gamma_2 \nabla_1 \nabla_2 + \gamma_2 \gamma_1 \nabla_2 \nabla_1 + \gamma_2^2 \nabla_2^2\right) \\
&= -\left(\nabla_1^2 + \gamma_1 \gamma_2 \nabla_1 \nabla_2 + \left(-\gamma_1 \gamma_2\right) \nabla_1 \nabla_2 + \nabla_2^2\right) \\
&= -\left(\nabla_1^2 + \nabla_2^2\right) = \nabla_1^* \nabla_1 + \nabla_2^* \nabla_2.
\end{aligned}$$

We have used the fact $\nabla_1 \nabla_2 = \nabla_2 \nabla_1$ and this is only true in the case where $A = 0$. (In the general case, they differ by $i(\partial_1 A_2 - \partial_2 A_1)$.) Now clearly $\nabla_\mu^* \nabla_\mu \psi = 0$ imply $D^* D\psi = 0$. To prove the other direction, consider

$$\begin{aligned}
\langle D^* D\psi, \psi \rangle &= \langle (\nabla_1^* \nabla_1 + \nabla_2^* \nabla_2)\psi, \psi \rangle \\
&= \langle \nabla_1^* \nabla_1 \psi, \psi \rangle + \langle \nabla_2^* \nabla_2 \psi, \psi \rangle \\
&= \langle \nabla_1 \psi, \nabla_1 \psi \rangle + \langle \nabla_2 \psi, \nabla_2 \psi \rangle \\
&= \|\nabla_1 \psi\|^2 + \|\nabla_2 \psi\|^2.
\end{aligned}$$

Thus $D^* D\psi = 0$ implies $\nabla_\mu \psi = 0$, which in turn implies that $\nabla_\mu^* \nabla_\mu \psi = 0$.

Now we prove the third equivalence. It's clear that $\nabla_\mu \psi = 0$ implies $\nabla_\mu^* \nabla_\mu \psi = 0$. For the other direction, we have

$$\left\langle \psi, \nabla_\mu^* \nabla_\mu \psi \right\rangle = \langle \nabla_\mu \psi, \nabla_\mu \psi \rangle = \|\nabla_\mu \psi\|^2.$$

Thus $\nabla_\mu^* \nabla_\mu \psi = 0$ imply $\nabla_\mu \psi = 0$. □

We're now ready to compute the index of D. Since $A = 0$, ∇_μ are just ∂_μ. Therefore,

$$D\psi = 0 \Leftrightarrow \partial_\mu \psi = 0$$

for $\mu = 1, 2$. In other words,

$$\ker D = \mathbb{C}^2.$$

To get the index of D, we need to apply the projection operators $P_\pm$ to $\ker D$. Because $\gamma_3 \neq \pm I$ ($\mathrm{tr}\,(\gamma_3) = 0$), $P_\pm(\ker D) \neq 0$. Thus

$$\dim(\ker D)_+ = \dim(\ker D)_- = 1.$$

Hence,

$$\mathrm{Index}\,(D) = 0.$$

3 Discretization of the Atiyah–Singer Index Theorem

3.1 Motivations

In this section we give several reasons for developing a discrete version of the Atiyah–Singer index theorem. One of the main motivations comes from gauge theories of particle physics. Various subatomic particles like protons, neutrons etc. are made up of quarks bound together by the strong nuclear force, which is described by a gauge theory called Quantum Chromodynamics (QCD). The index theorem plays an important role in explaining why certain subnuclear particles have a larger mass than expected. The calculation of particle masses in the theory requires a discrete lattice formulation of QCD in order to be implemented on a computer. Therefore we need to have index theorem in discrete setting to get correct masses from lattice QCD calculations.

From the mathematical point of view, discretization is a technique which appears frequently. For instance, to solve PDE numerically, we need to discretize the domain and the differential operators. In topology, we can decompose topological spaces into discrete "cells" and then can describe some topological invariants combinatorially (the Euler characteristic is a famous example). If a discrete version of the AS index theorem is established, the usual AS index theorem will be recovered in the continuum limit and that gives us a new proof of the AS index theorem.

In the remaining sections, we'll develop the discrete counterparts of gauge fields, topological charge, index etc. required for the formulation of the discrete index theorem. There may be several ways to discretize a continuous entity. Nevertheless, they should all reproduce their continuous counterparts in the continuum limit.

3.2 Lattice Gauge Fields

A lattice on $\mathbb{R}^2$ is the following collection of points

$$L = \left\{(n_1 a, n_2 a) \in \mathbb{R}^2 \colon (n_1, n_2) \in \mathbb{Z}^2,\ a = 1/N\right\}.$$

Here N is some positive integer. We call an element of the lattice a *lattice site* and a the *lattice spacing*. The line segment that connects x and $x + ae_\mu$ is called a *link*. The number of lattice sites on each unit interval is

$$\frac{1}{a} + 1 = N + 1.$$

Here we consider the vector space V of functions $L \to \mathbb{C}^2$ satisfying the twisted periodicity condition:

$$\begin{cases} f(x_1+1, x_2) = f(x_1, x_2), \\ f(x_1, x_2+1) = e^{i\theta(x_1)} f(x_1, x_2) \end{cases} \tag{3.1}$$

for $(x_1, x_2) \in L$. Similar to the continuum case, the function θ has to satisfy the consistency requirement

$$\begin{aligned} f(x_1+1, x_2+1) &= f(x_1, x_2+1) = e^{i\theta(x_1)} f(x_1, x_2) \\ &= e^{i\theta(x_1+1)} f(x_1+1, x_2) = e^{i\theta(x_1+1)} f(x_1, x_2). \end{aligned}$$

We deduce that

$$\theta(x_1+1) = \theta(x_1) + 2\pi Q(x_1).$$

Note that in the discrete case, the integer Q doesn't have to be a constant since the notion of continuity no longer applies for the function f. We can turn V into an inner-product space by

$$\langle f, g \rangle = a^2 \sum_{x \in \Gamma} g(x)^* f(x), \quad \text{for } f, g \in V.$$

where

$$\Gamma = \{(n_1 a, n_2 a) \in L \colon 0 \le n_1, n_2 < N\}.$$

To justify our definition of the inner product, recall that in the continuum case we have

$$\langle f, g \rangle = \int_{[0,1]^2} g(x)^* f(x) dx = \lim_{N \to \infty} a^2 \sum_{0 \le n_1, n_2 < N} g(n_1 a, n_2 a)^* f(n_1 a, n_2 a)$$

for continuous functions f and g. The discrete inner product satisfies the following property

$$\langle f, g \rangle = a^2 \sum_{x \in \Gamma} g(x)^* f(x) = a^2 \sum_{x \in \Gamma} g(x \pm a e_\mu)^* f(x \pm a e_\mu), \quad \mu = 1, 2, \tag{3.2}$$

which can be checked by direct computation. Contrary to the continuum case, the vector space V is no longer infinite dimensional since a function on the lattice sites is determined by its values on Γ. An orthonormal basis for V can be given by

$$\left\{ \frac{\delta_x^i}{a} \colon x \in \Gamma \text{ and } i = 1, 2 \right\},$$

where δ_x^i is a function on the lattice sites given by

$$\delta_x^i(y) = \delta_{xy} e_i = \begin{cases} 0 & \text{if } y \neq x, \\ e_i & \text{if } y = x \end{cases}$$

for $y \in \Gamma$ and extend to the whole lattice using the boundary conditions. (Here e_i denotes the standard basis element of $\mathbb{C}^2$.) Thus the dimension of V is $2N^2$.

In the lattice we replace the partial derivative operators by finite-difference operators:

$$\partial_\mu f(x) \to \frac{1}{2a}[f(x+ae_\mu) - f(x-ae_\mu)].$$

However, similar to the continuum, the twisted periodicity condition is not preserved. For instance

$$\begin{aligned}\frac{1}{2a}&[f(x_1+a,x_2+1) - f(x_1-a,x_2+1)] \\ &= \frac{1}{2a}[e^{i\theta(x_1+a)}f(x_1+a,x_2) - e^{i\theta(x_1-a)}f(x_1-a,x_2)],\end{aligned}$$

which is different from

$$\frac{1}{2a}e^{i\theta(x_1)}[f(x_1+a,x_2) - f(x_1-a,x_2)].$$

We need to introduce a lattice version of the gauge field A. By a *lattice gauge field* U we mean a collection of U(1)-valued functions on the links of the lattice. The twisted periodicity condition will impose certain conditions on U, as we're going to see below. We'll use the notation

Figure 3.1: The link variables $U_\mu(x)$ and $U_\mu(x)^{-1}$

$$U_\mu(x) \leftrightarrow U(x, x+ae_\mu)$$

to denote the element of $\mathrm{U}(1)$ associated with the link that goes from $x+ae_\mu$ to x. The element of $\mathrm{U}(1)$ associated with a link that goes from x to $x+ae_\mu$ is naturally defined to be $U_\mu(x)^{-1}$, hence the notation

$$U_\mu(x)^{-1} \leftrightarrow U(x+ae_\mu, x).$$

We also call these U_μ *link variables* (Figure 3.1). Now the finite-difference operators are replaced by

$$\nabla_\mu f(x) \colon = \frac{1}{2a}(U_\mu(x)f(x+ae_\mu) - U_\mu(x-ae_\mu)^{-1}f(x-ae_\mu)). \tag{3.3}$$

We can write ∇_μ as

$$\nabla_\mu = \frac{1}{2}\left(\nabla_\mu^+ + \nabla_\mu^-\right),$$

where

$$\nabla_\mu^+ f(x) = \frac{1}{a}(U_\mu(x) f(x + ae_\mu) - f(x)),$$
$$\nabla_\mu^- f(x) = \frac{1}{a}(f(x) - U_\mu(x - ae_\mu)^{-1} f(x - ae_\mu)),$$

We want the ∇_μ to map V to itself, i.e.,

$$\nabla_\mu f(x_1 + 1, x_2) = \nabla_\mu f(x_1, x_2),$$
$$\nabla_\mu f(x_1, x_2 + 1) = e^{i\theta(x_1)} \nabla_\mu f(x_1, x_2)$$

for all $f \in V$ and $\mu = 1, 2$. Proceed analogously as in the continuum case, we obtain the following requirement for the lattice gauge field:

$$\begin{cases} U_1(x_1 + 1, x_2) = U_1(x_1, x_2), \\ U_1(x_1, x_2 + 1) = e^{i\theta(x_1)} U_1(x_1, x_2) e^{-i\theta(x_1 + a)}, \\ U_2(x_1 + 1, x_2) = U_2(x_1, x_2), \\ U_2(x_1, x_2 + 1) = U_2(x_1, x_2). \end{cases} \tag{3.4}$$

If we define $\theta(x_1) = 2\pi Q x_1$ (note that now x_1 is a lattice point) for some integer Q, then a lattice gauge field can be given by

$$\begin{cases} U_1(x_1, x_2) = e^{-i2\pi Q x_2 / N}, \\ U_2(x_1, x_2) = 1. \end{cases}$$

for $(x_1, x_2) \in L$. Observe that $U_\mu(x)$ is essentially the exponential of $A_\mu(x)$. Of course the choice of U is not unique, we can replace U by

$$\begin{cases} \widehat{U}_1(x_1, x_2) = e^{-i2\pi Q x_2 / N} e^{iR(x)}, \\ \widehat{U}_2(x_1, x_2) = e^{iR(x)}, \end{cases}$$

where $R(x)$ is a real-valued function on the lattice which is periodic in both directions.

3.3 Relation to Smooth Gauge Fields

A lattice gauge field can be obtained from a continuum gauge field by taking *transcript*. Specifically, U_μ is given by

$$U_\mu(x) = e^{i\alpha_\mu(x)}, \text{ where } \alpha_\mu(x) = \int_0^a A_\mu(x + (a - t)e_\mu) dt.$$

Thus $\alpha_\mu(x)$ is just the integral of A_μ over the link that goes from $x + ae_\mu$ to x. When we reverse the direction of the link, the integral will pick up a minus sign. That explains our notation:

$$U(x, x + ae_\mu) = U(x + ae_\mu, x)^{-1}.$$

It can be checked that $\{U_\mu\}$ satisfy the conditions for a lattice gauge field. Indeed,

$$\begin{aligned}
U_1(x_1, x_2+1) &= \exp\left(i\int_0^a A_1(x_1+a-t, x_2+1)dt\right)\\
&= \exp\left(i\int_0^a (A_1(x_1+a-t, x_2) - \theta'(x_1+a-t))dt\right)\\
&= \exp\left(i\int_0^a A_1(x+(a-t)e_1)dt\right)\exp\left(-i\int_0^a \theta'(x_1+a-t)dt\right)\\
&= U_1(x_1,x_2)e^{i(\theta(x_1)-\theta(x_1+a))}.
\end{aligned}$$

The other conditions can be shown similarly. Since A_μ are smooth, we can use the Taylor expansion around the point x:

$$A_\mu(x+(a-t)e_\mu) = A_\mu(x) + \frac{\partial A_\mu}{\partial x_\mu}(x)(a-t) + \mathcal{O}((a-t)^2).$$

Hence

$$\int_0^a A_\mu(x+(a-t)e_\mu)dt = aA_\mu(x) + \frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) + \mathcal{O}(a^3).$$

Now exponentiate both sides we obtain an expansion for $U_\mu(x)$

$$\begin{aligned}
U_\mu(x) &= \exp\left(i\int_0^a A_\mu(x+(a-t)e_\mu)dt\right)\\
&= 1 + i\left(aA_\mu(x) + \frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) + \mathcal{O}(a^3)\right)\\
&\quad - \frac{1}{2}\left(aA_\mu(x) + \frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) + \mathcal{O}(a^3)\right)^2 + \cdots\\
&= 1 + iaA_\mu(x) + i\frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) - \frac{a^2}{2}A_\mu^2(x) + \mathcal{O}(a^3).
\end{aligned}$$

Thus up to first order,

$$U_\mu(x) = 1 + iaA_\mu(x) + \mathcal{O}(a^2).$$

Similarly,

$$U_\mu^{-1}(x) = 1 - iaA_\mu(x) + \mathcal{O}(a^2).$$

When a lattice gauge field U is the transcript of some continuum gauge field A, we'll show that our discrete operators approach its continuum analogues $\partial_\mu + iA_\mu$ as a tends to 0 (N tends to ∞). Indeed for a smooth function f, using the expansion of U_μ we obtain

$$\begin{aligned}
\nabla_\mu f(x) &= \frac{1}{2a}\left(U_\mu(x)f(x+ae_\mu) - U_\mu^{-1}(x-ae_\mu)f(x-ae_\mu)\right)\\
&= \frac{1}{2a}(1+iaA_\mu(x)+\mathcal{O}(a^2))f(x+ae_\mu)\\
&\quad - \frac{1}{2a}(1-iaA_\mu(x-ae_\mu)+\mathcal{O}(a^2))f(x-ae_\mu)\\
&= \frac{f(x+ae_\mu)-f(x-ae_\mu)}{2a} + \frac{i}{2}\left(A_\mu(x)f(x+ae_\mu)+A_\mu(x-ae_\mu)f(x-ae_\mu)\right)\\
&\quad + \mathcal{O}(a)(f(x+ae_\mu)-f(x-ae_\mu)).
\end{aligned}$$

Thus when $a \to 0$,

$$\nabla_\mu f(x) \to \partial_\mu f(x) + iA_\mu(x)f(x).$$

3.4 Gauge Transformations on the Lattice

We also have the concept of gauge transformations for lattice gauge fields. Similar to the continuum case, a gauge transformation is a function from the lattice sites to $\mathrm{U}(1)$, which we can write as $e^{i\phi(x)}$. Our vector space will be transformed into

$$\widetilde{V} = \left\{e^{i\phi}f\colon f \in V\right\}.$$

The lattice gauge field U will be transformed into $\widetilde{U}$ according to:

$$\widetilde{\nabla}_\mu(e^{i\phi}f) = e^{i\phi}\nabla_\mu f.$$

The left hand side is given by

$$\frac{1}{2a}\left[\widetilde{U}_\mu(x)e^{i\phi(x+ae_\mu)}f(x+ae_\mu) - \widetilde{U}_\mu(x-ae_\mu)^{-1}e^{i\phi(x-ae_\mu)}f(x-ae_\mu)\right],$$

and the right hand side is given by

$$\frac{1}{2a}e^{i\phi(x)}\left[U_\mu(x)f(x+ae_\mu) - U_\mu(x-ae_\mu)^{-1}f(x-ae_\mu)\right].$$

By comparing the left hand side and the right hand side we obtain

$$\widetilde{U}_\mu(x) = e^{i\phi(x)}U_\mu(x)e^{-i\phi(x+ae_\mu)}$$

for all lattice sites x and that is how a lattice gauge field is transformed under a gauge transformation.

In the case where U is the lattice transcript of some smooth gauge field A and $e^{i\phi}$ is a gauge transformation in the continuum, we'll show that the lattice transcript of $\widetilde{A}$ is precisely the lattice gauge field $\widetilde{U}$ obtained from U by the gauge transformation $e^{i\phi}$ (the restriction of $e^{i\phi}$ to the lattice sites). Indeed, the lattice transcript of $\widetilde{A}$ is given by

$$\begin{aligned}
e^{i\widetilde{\alpha}_\mu(x)} &= \exp\left(i\int_0^a \widetilde{A}_\mu(x+(a-t)e_\mu)dt\right)\\
&= \exp\left(i\int_0^a \left[A_\mu(x+(a-t)e_\mu) - \partial_\mu\phi(x+(a-t)e_\mu)\right]dt\right)\\
&= \exp\left(i\int_0^a A_\mu(x+(a-t)e_\mu)dt\right)\exp\left(-i\int_0^a \partial_\mu\phi(x+(a-t)e_\mu)dt\right)\\
&= U_\mu(x)\exp\left(-i\int_0^a \partial_\mu\phi(x+(a-t)e_\mu)dt\right).
\end{aligned}$$

Now the term

$$\int_0^a \partial_\mu\phi(x+(a-t)e_\mu)dt = -\int_l d\phi = \phi(x+ae_\mu) - \phi(x),$$

where l is the line segment from $x+ae_\mu$ to x and the second equality follows from the fundamental theorem of calculus. Thus

$$e^{i\widetilde{\alpha}_\mu(x)} = U(x)e^{i[\phi(x)-\phi(x+ae_\mu)]} = \widetilde{U}_\mu(x).$$

3.5 Naive Index

We can define the Dirac operator in the discrete case as

$$D = \gamma_1\nabla_1 + \gamma_2\nabla_2.$$

Here ∇_μ are our finite-difference operators (3.3) and the gamma matrices are the same as in the continuum case. As operators on V, the gamma matrices are hermitian because

$$\begin{aligned}
\langle \gamma_\mu f, g\rangle &= a^2\sum_{x\in\Gamma} g(x)^*\gamma_\mu f(x) = a^2\sum_{x\in\Gamma} g(x)^*\gamma_\mu^* f(x)\\
&= a^2\sum_{x\in\Gamma}(\gamma_\mu g(x))^* f(x) = \langle f, \gamma_\mu g\rangle.
\end{aligned}$$

For any unit vector $\psi \in V$ we have

$$\|\gamma_\mu\psi\|^2 = \langle\gamma_\mu\psi, \gamma_\mu\psi\rangle = \langle\psi, \gamma_\mu^*\gamma_\mu\psi\rangle = \langle\psi,\psi\rangle = 1.$$

Thus $\|\gamma_\mu\| = 1$. Since the vector space V is finite-dimensional, $\ker D$ is finite-dimensional. The (analytical) index of D can be defined verbatim as in Section 2.6 of Chapter 2. Thus

$$\mathrm{Index}\,(D) = \mathrm{tr}\,(\gamma_3\colon\ \ker V \to \ker V).$$

Naively, if the lattice gauge field is the transcript of some continuum gauge field, we hope that the index of D will agree with the index of its continuum analogue. However, it turns out that in the discrete case, the index of D defined in this way is always 0, as we are going to show below. First, we have (Theorem 1.2)

$$V = \ker D \oplus (\ker D)^{\perp}.$$

Recall that we have

$$\gamma_3 D = -D\gamma_3.$$

Thus $\ker D$ is invariant under γ_3. As an operator on V, γ_3 is hermitian. Moreover since $\gamma_3^2 = 1$ we also have

$$\|\gamma_3\| = 1.$$

Now for any $w \in (\ker D)^{\perp}$ and $z \in \ker D$,

$$\langle \gamma_3 w, z \rangle = \langle w, \gamma_3 z \rangle = 0,$$

where the last equality follows because $w \in (\ker D)^{\perp}$ and $\ker D$ is invariant under γ_3. We conclude that $(\ker D)^{\perp}$ is also invariant under γ_3. It follows that

$$\operatorname{tr}(\gamma_3 \colon V \to V) = \operatorname{tr}(\gamma_3 \colon \ \ker D \to \ker D) + \operatorname{tr}(\gamma_3 \colon (\ker D)^{\perp} \to (\ker D)^{\perp}).$$

We should check that D is anti-hermitian as in the continuum case, i.e.,

$$D^* = -D.$$

For this we need the following property of $\nabla_{\mu}^{\pm}$:

Proposition 3.1.

$$\left(\nabla_{\mu}^{\pm}\right)^* = -\nabla_{\mu}^{\mp}.$$

Proof. It suffices to show that

$$(\nabla_{\mu}^{+})^* = -\nabla_{\mu}^{-}.$$

Taking the adjoint of both sides we obtain

$$(\nabla_{\mu}^{-})^* = -(\nabla_{\mu}^{+})^{**} = -\nabla_{\mu}^{+}.$$

Now we have

$$\begin{aligned}
\left\langle f, -\nabla_{\mu}^{-} g \right\rangle &= a^2 \sum_{x\in\Gamma} (-\nabla_{\mu}^{-} g(x))^* f(x) \\
&= a^2 \sum_{x\in\Gamma} (U_{\mu}(x - ae_{\mu})^{-1} g(x - ae_{\mu}) - g(x))^* f(x) \\
&= a^2 \sum_{x\in\Gamma} (U_{\mu}(x - ae_{\mu}) g(x - ae_{\mu})^* - g(x)^*) f(x) \\
&= a^2 \left(\sum_{x\in\Gamma} U_{\mu}(x - ae_{\mu}) g(x - ae_{\mu})^* f(x) - \sum_{x\in\Gamma} g(x)^* f(x) \right).
\end{aligned}$$

Note that since $U(x) \in \mathrm{U}(1)$, we have $U(x)^{-1} = \overline{U(x)}$. Now it can be verified by direct computation that

$$\sum_{x\in\Gamma} U_\mu(x - ae_\mu) g(x - ae_\mu)^* f(x) = \sum_{x\in\Gamma} U_\mu(x) g(x)^* f(x + ae_\mu), \quad \mu = 1, 2.$$

Hence

$$\begin{aligned}
\left\langle f, -\nabla_\mu^- g\right\rangle &= a^2 \left(\sum_{x\in\Gamma} U_\mu(x) g(x)^* f(x + ae_\mu) - \sum_{x\in\Gamma} g(x)^* f(x) \right) \\
&= a^2 \sum_{x\in\Gamma} g(x)^* \nabla_\mu^+ f(x) \\
&= \left\langle \nabla_\mu^+ f, g \right\rangle .
\end{aligned}$$

Therefore $(\nabla_\mu^+)^* = -\nabla_\mu^-$. □

Proposition 3.2.

$$D^* = -D.$$

Proof. Recall that

$$\nabla_\mu = \frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-).$$

Thus

$$\nabla_\mu^* = \frac{1}{2}(-\nabla_\mu^- - \nabla_\mu^+) = -\nabla_\mu$$

for $\mu = 1, 2$. It follows that

$$D^* = \sum_\mu \gamma_\mu^* \nabla_\mu^* = -\sum_\mu \gamma_\mu \nabla_\mu = -D.$$

□

Now take any $v \in (\ker D)^\perp$ and $w \in \ker D$. Since D is anti-hermitian,

$$\langle Dv, w\rangle = \langle v, D^* w\rangle = -\langle v, Dw\rangle = \langle v, 0\rangle = 0.$$

Thus $Dv \in (\ker D)^\perp$. In other words, $(\ker D)^\perp$ is invariant under D. More importantly,

Proposition 3.3. *The operator $D\colon (\ker D)^\perp \to (\ker D)^\perp$ is invertible.*

Proof. Since $(\ker D)^\perp$ is finite-dimensional, $D|_{(\ker D)^\perp}$ is invertible if and only if its kernel is trivial. Take any $w \in \ker \left(D|_{(\ker D)^\perp} \right)$, then

$$D|_{(\ker D)^\perp}\, w = Dw = 0.$$

Thus

$$w \in \ker D \cap (\ker D)^\perp.$$

Hence $w = 0$. □

Proposition 3.4.

$$\operatorname{tr}(\gamma_3\colon (\ker D)^{\perp} \to (\ker D)^{\perp}) = 0.$$

Proof. To avoid cumbersome notations, in what follows we abbreviate $D|_{(\ker D)^{\perp}}$ to D. We have:

$$\begin{aligned}\operatorname{tr}(\gamma_3\colon (\ker D)^{\perp} \to (\ker D)^{\perp}) &= \operatorname{tr}(\gamma_3 D D^{-1}) = \operatorname{tr}(D^{-1}\gamma_3 D)\\ &= -\operatorname{tr}(D^{-1}D\gamma_3\colon (\ker D)^{\perp} \to (\ker D)^{\perp})\\ &= -\operatorname{tr}(\gamma_3\colon (\ker D)^{\perp} \to (\ker D)^{\perp}),\end{aligned}$$

where we've used $\gamma_3 D = -D\gamma_3$ in the second equality. Thus

$$\operatorname{tr}(\gamma_3\colon (\ker D)^{\perp} \to (\ker D)^{\perp}) = 0.$$

□

We conclude that

$$\operatorname{tr}(\gamma_3\colon V \to V) = \operatorname{tr}(\gamma_3\colon \ker D \to \ker D).$$

To finish up, we need to show that

$$\operatorname{tr}(\gamma_3\colon V \to V) = 0.$$

This is a simple exercise in linear algebra. Recall that V has the following orthonormal basis

$$\left\{\frac{1}{a}\delta_x^1, \frac{1}{a}\delta_x^2\colon x \in \Gamma\right\},$$

where δ_x^i is a function on the lattice sites given by

$$\delta_x^i(y) = \delta_{xy}e_i = \begin{cases} 0 & \text{if } y \neq x,\\ e_i & \text{if } y = x\end{cases}$$

for $y \in \Gamma$ and extend to the whole lattice using the boundary conditions. Then it's not hard to verify that with respect to this basis, γ_3 is a $2N^2 \times 2N^2$ (here N is the number of lattice sites) block-diagonal matrix where each block is the matrix γ_3. It follows that

$$\operatorname{tr}(\gamma_3\colon V \to V) = N^2\operatorname{tr}(\gamma_3\colon \mathbb{C}^2 \to \mathbb{C}^2) = 0.$$

3.6 Spectral Flow

In Section 3.5 we see that no matter what the index of the continuum Dirac operator is, the index of its discrete analogue is always zero. Thus the discrete Dirac operator doesn't give any meaningful information about the continuum and that prompts a need for a new definition of the index. This section aims to motivate our new definition of the index in the lattice setting. The treatment is not entirely rigorous and intuitive understanding is emphasized.

For our new definition of the index, we first need to discuss the notion of the *spectral flow*. Let $H(m)$ be a family of self-adjoint operators depending smoothly on some real parameter m. (For concreteness, we can take a basis for the vector space and hence $H(m)$ is a family of matrices whose entries are smooth functions of m.) Assume we have k eigenvalues $\lambda_1(m), ..., \lambda_k(m)$ which depend smoothly on m. (The characteristic polynomial of $H(m)$ is a polynomial whose coefficients are functions of m and we can solve that polynomial to get the eigenvalues depending on m. These are smooth real-valued functions because the operators are self-adjoint.) Intuitively, the *spectral flow* of $H(m)$ in some interval (a, b) is the difference between the number of eigenvalues that cross the x-axis with positive slope and the number of eigenvalues that cross the x-axis with negative slope, counting multiplicity. (There's some ambiguity regarding the word multiplicity that we use here. For self-adjoint operators the algebraic multiplicity is the same as the geometric multiplicity and it can be shown that the multiplicity of $\lambda(m)$ is constant except for a finite number of points [13].)

The family of operators of our interest is

$$H(m) = \gamma_3(D - m), \quad m \in \mathbb{R},$$

where D is our naive Dirac operator as defined in Section 3.5. Recall that D is anti-hermitian, i.e.,

$$D^* = -D.$$

Similar to hermitian operators, D can be represented as a diagonal matrix with respect to some orthonormal basis of V. To see why, consider the operator iD, then

$$(iD)^* = -iD^* = iD.$$

Thus iD is hermitian and so there exists an orthonormal basis for V consisting of eigenvectors of iD. It follows that D can be expressed as a diagonal matrix with respect to the same basis. If μ is an eigenvalue of D, then for some nonzero vector $v \in V$,

$$\mu = \langle Dv, v\rangle = \langle v, D^*v\rangle = -\langle v, Dv\rangle = -\overline{\mu}.$$

Thus an eigenvalue of D is purely imaginary, i.e., it is of the form $i\lambda$ for some real number λ. Moreover,

Proposition 3.5. *The eigenvalues of D are symmetric about the real axis.*

Proof. We have the relation

$$\gamma_3 D = -D\gamma_3.$$

Now suppose λ is an eigenvalue of D, i.e., there exists some nonzero vector $v \in V$ such that

$$Dv = \lambda v.$$

Hence
$$D(\gamma_3 v) = -\gamma_3(Dv) = -\gamma_3(\lambda v) = -\lambda(\gamma_3 v).$$
It follows that $\gamma_3 v$ is an eigenvector of D corresponding to $-\lambda$. Since γ_3 is invertible, the eigenspaces for λ and $-\lambda$ are isomorphic and thus λ and $-\lambda$ have the same multiplicities. □

For any $m \in \mathbb{R}$, we have
$$H(m)^2 = \gamma_3(D-m)\gamma_3(D-m) = (-D-m)(D-m) = -D^2 + m^2,$$
where we've used $\gamma_3 D = -D\gamma_3$ and $\gamma_3^2 = 1$ in the second equality. Choose an orthonormal basis for V consisting of eigenvectors of D, then for any element ψ in that basis,
$$H(m)^2\psi = (-D^2 + m^2)\psi = (-(i\lambda)^2 + m^2)\psi = (\lambda^2 + m^2)\psi.$$
Thus the eigenvalues of $H(m)^2$ are $\lambda^2 + m^2$ for some real numbers λ. Specifically, these are the real numbers that satisfy
$$Dv = i\lambda v$$
for some nonzero vector v. Note that different λ give us different eigenvalues of $H(m)^2$. Thus the possible eigenvalues of $H(m)$ are $\pm\sqrt{\lambda^2 + m^2}$. We are interested in those eigenvalues that cross the x-axis and that occur when
$$\lambda^2 + m^2 = 0.$$
Since m and λ are real, it is equivalent to $\lambda = m = 0$. It follows that the only possible eigenvalues that cross the x-axis are
$$f(m) = \pm m$$
and the crossings only occur at the origin. We now compute the spectral flow of $H(m)$ in the interval $(-\infty, \infty)$. From the above argument, for a fixed $m \neq 0$,
$$H(m)\psi = \pm m\psi$$
implies that
$$H(m)^2\psi = m^2\psi$$
and so $\lambda = 0$ and $\psi \in \ker D$. Moreover, for $\psi_+ \in (\ker D)_+$,
$$H(m)\psi_+ = \gamma_3(D-m)\psi_+ = -m\gamma_3\psi_+ = -m\psi_+$$
and for $\psi_- \in (\ker D)_-$,
$$H(m)\psi_- = \gamma_3(D-m)\psi_- = -m\gamma_3\psi_- = m\psi_-.$$
Hence the multiplicity of m is $\dim(\ker D)_-$ and the multiplicity of $-m$ is $\dim(\ker D)_+$. It follows that
$$\operatorname{Index}(D) = \dim(\ker D)_+ - \dim(\ker D)_-$$

is the difference between the number of eigenvalues of $H(m)$ crossing the x-axis with negative slope and the number of eigenvalues of $H(m)$ crossing the x-axis with positive slope, which is precisely minus the spectral flow of $H(m)$ in $(-\infty, \infty)$.

In our case of course

$$\dim(\ker D)_+ = \dim(\ker D)_-$$

and the spectral flow of $H(m)$ is zero because the index of D is always zero. Hence besides the elements of $(\ker D)_\pm$, which we call *smooth solutions*, that give us the solutions to $D = 0$ in the continuum (when we take $a \to 0$), there are elements in $(\ker D)_\pm$ that don't correspond to any solutions of $D = 0$ in the continuum and we call those *rough solutions*. To get a meaningful definition of the index in the lattice setting, we need a way to separate the rough and smooth solutions. We do that by introducing a modified version of our naive Dirac operator, the so called *Wilson-Dirac operator*:

$$D_w = D + \frac{a}{2}\Delta.$$

Here Δ is a lattice version of the second derivative:

$$\Delta = \sum_{\mu=1,2} \Delta_\mu, \quad \Delta_\mu = -\nabla_\mu^+ \nabla_\mu^-.$$

The explicit formula for Δ_μ is given by

$$\Delta_\mu f(x) = \frac{1}{a^2}(-U_\mu(x) f(x + ae_\mu) + 2f(x) - U_\mu(x - ae_\mu)^{-1} f(x - ae_\mu)).$$

The operator Δ has several important properties.

Proposition 3.6.

$$\Delta_\mu = -\nabla_\mu^+ \nabla_\mu^- = -\nabla_\mu^- \nabla_\mu^+ = \frac{1}{a}(\nabla_\mu^- - \nabla_\mu^+).$$

Proof. These equalities follow from direct computation. We have:

$$\begin{aligned}
\nabla_\mu^+(\nabla_\mu^- f)(x) &= \frac{1}{a}(U_\mu(x)\nabla_\mu^- f(x + ae_\mu) - \nabla_\mu^- f(x)) \\
&= \frac{1}{a}\left[U_\mu(x)\left(\frac{1}{a}\left[f(x + ae_\mu) - U_\mu(x)^{-1} f(x)\right]\right)\right. \\
&\qquad \left. - \left(\frac{1}{a}\left[f(x) - U_\mu(x - ae_\mu)^{-1} f(x - ae_\mu)\right]\right)\right] \\
&= \frac{1}{a^2}\left[U_\mu(x) f(x + ae_\mu) - 2f(x) + U_\mu(x - ae_\mu)^{-1} f(x - ae_\mu)\right],
\end{aligned}$$

and

$$
\begin{aligned}
\nabla_\mu^-(\nabla_\mu^+ f)(x) &= \frac{1}{a}(\nabla_\mu^+ f(x) - U_\mu(x-ae_\mu)^{-1}\nabla_\mu^+ f(x-ae_\mu)) \\
&= \frac{1}{a}\left[\left(\frac{1}{a}\left[U_\mu(x)f(x+ae_\mu) - f(x)\right]\right)\right. \\
&\qquad \left. - U_\mu(x-ae_\mu)^{-1}\left(\frac{1}{a}\left[U_\mu(x-ae_\mu)f(x) - f(x-ae_\mu)\right]\right)\right] \\
&= \frac{1}{a^2}\left[U_\mu(x)f(x+ae_\mu) - 2f(x) + U_\mu(x-ae_\mu)^{-1}f(x-ae_\mu)\right].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{a}(\nabla_\mu^+ - \nabla_\mu^-)f(x) &= \frac{1}{a}\left(\frac{1}{a}\left[U_\mu(x)f(x+ae_\mu) - f(x)\right]\right. \\
&\qquad \left. - \frac{1}{a}\left[f(x) - U_\mu(x-ae_\mu)^{-1}f(x-ae_\mu)\right]\right) \\
&= \frac{1}{a^2}\left[U_\mu(x)f(x+ae_\mu) - 2f(x) + U_\mu(x-ae_\mu)^{-1}f(x-ae_\mu)\right].
\end{aligned}
$$

□

Proposition 3.7. *The operator Δ is positive, i.e., $\|\Delta\| \geq 0$. Here $\|\ \|$ denotes the operator norm (Section 1.3). In particular, Δ is hermitian.*

Proof. Proceed by definition of positive operators:

$$\langle \Delta_\mu f, f\rangle = \left\langle -\nabla_\mu^+\nabla_\mu^- f, f\right\rangle = \left\langle \nabla_\mu^- f, (-\nabla_\mu^+)^* f\right\rangle = \left\langle \nabla_\mu^- f, \nabla_\mu^- f\right\rangle \geq 0,$$

where we again use Proposition 3.1. Since sums of positive operators are positive, Δ is positive. □

Unlike the naive Dirac operator D, D_w is no longer anti-hermitian:

$$D_w^* = \left(D + \frac{a}{2}\Delta\right)^* = -D + \frac{a}{2}\Delta.$$

And the relation $\gamma_3 D = -D\gamma_3$ becomes

$$\gamma_3 D_w = \gamma_3\left(D + \frac{a}{2}\Delta\right) = -D\gamma_3 + \frac{a}{2}\Delta\gamma_3 = D_w^*\gamma_3.$$

Note that Δ and γ_3 commute since Δ does not contain any gamma matrices. This equality means that $\ker D_w$ is no longer invariant under γ_3 and hence we can't define the index of D_w in terms of chirality. The positivity of Δ has another consequence: suppose m is a real eigenvalue of D_w, i.e.,

$$D_w\psi = m\psi$$

for some nonzero vector $\psi \in V$ such that $\|\psi\| = 1$. Then

$$\langle a\Delta\psi, \psi\rangle = \langle (D_w + D_w^*)\psi, \psi\rangle = \langle D_w\psi, \psi\rangle + \langle \psi, D_w\psi\rangle = 2m.$$

Since Δ is positive, we conclude that $m \geq 0$. Thus D_w has only non-negative real eigenvalues.

3.7 Hermitian Wilson-Dirac Operator

Similar to the Dirac operator we consider the following family of operators:

$$H_w(m) = \gamma_3(D_w - m), \quad m \in \mathbb{R}.$$

Using the relation $\gamma_3 D_w = D_w^* \gamma_3$, we have

$$H_w(m)^* = (D_w^* - m)\gamma_3^* = \gamma_3(D_w - m) = H_w(m).$$

Thus $H_w(m)$ is hermitian and so all its eigenvalues are real. We are interested in the eigenvalue flow of $H_w(m)$. First, if

$$H_w(m)\psi = 0$$

for some nonzero vector $\psi \in V$, then

$$D_w \psi = m\psi.$$

So m is an eigenvalue of D_w. Thus the eigenvalues of $H_w(m)$ cross the x-axis at real eigenvalues of D_w. Since real eigenvalues of D_w are non-negative, eigenvalue crossings of $H_w(m)$ can only occur when $m \geq 0$. Another important property of the eigenvalues of $H_w(m)$ is given by the next proposition.

Proposition 3.8. *Let $\lambda(m)$ be an eigenvalue of $H_w(m)$, then*

$$|\lambda'(m)| \leq 1.$$

Proof. We have

$$H(m)\psi(m) = \lambda(m)\psi(m),$$

where $\psi(m)$ also depends smoothly on m and $\|\psi(m)\| = 1$ (think of $\psi(m)$ as a vector whose entries are smooth functions of m). Thus

$$\langle H(m)\psi(m), \psi(m)\rangle = \lambda(m)\langle \psi(m), \psi(m)\rangle = \lambda(m).$$

Now we can differentiate $\lambda(m)$:

$$\begin{aligned}
\lambda'(m) &= \lim_{h\to 0} \frac{1}{h}\left(\langle H(m+h)\psi(m+h), \psi(m+h)\rangle - \langle H(m)\psi(m), \psi(m)\rangle\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\langle H(m+h)\psi(m+h), \psi(m+h)\rangle - \langle H(m+h)\psi(m+h), \psi(m)\rangle\right. \\
&\qquad + \langle H(m+h)\psi(m+h), \psi(m)\rangle - \langle H(m)\psi(m+h), \psi(m)\rangle \\
&\qquad \left. + \langle H(m)\psi(m+h), \psi(m)\rangle - \langle H(m)\psi(m), \psi(m)\rangle\right) \\
&= \left\langle H(m)\psi(m), \psi'(m)\right\rangle + \left\langle H'(m)\psi(m), \psi(m)\right\rangle + \left\langle H(m)\psi'(m), \psi(m)\right\rangle.
\end{aligned}$$

Now note that

$$\left\langle H(m)\psi'(m), \psi(m)\right\rangle = \left\langle \psi'(m), H(m)\psi(m)\right\rangle = \lambda(m)\left\langle \psi'(m), \psi(m)\right\rangle.$$

Thus $\lambda'(m)$ becomes

$$\lambda'(m) = \lambda(m)\left(\left\langle \psi(m), \psi'(m)\right\rangle + \left\langle \psi'(m), \psi(m)\right\rangle\right) + \left\langle H'(m)\psi(m), \psi(m)\right\rangle.$$

Now we have

$$\frac{d}{dm}\|\psi(m)\|^2 = \frac{d}{dm}\langle \psi(m), \psi(m)\rangle = \left\langle \psi'(m), \psi(m)\right\rangle + \left\langle \psi(m), \psi'(m)\right\rangle.$$

Since $\|\psi(m)\|^2 = 1$, its derivative is 0. Thus

$$\lambda'(m) = \left\langle H'(m)\psi(m), \psi(m)\right\rangle.$$

Moreover,

$$H'(m) = \frac{d}{dm}\gamma_3(D_w - m) = -\gamma_3.$$

Consequently,

$$\lambda'(m) = -\langle \gamma_3\psi(m), \psi(m)\rangle.$$

Now by Cauchy–Schwarz inequality,

$$|\lambda'(m)| \le \|\gamma_3\psi(m)\|\|\psi(m)\| = \|\gamma_3\psi(m)\| \le 1$$

since

$$\|\gamma_3\psi(m)\| \le \|\gamma_3\|\|\psi(m)\| = 1$$

(recall that $\|\gamma_3\| = 1$). □

As was mentioned previously, the introduction of Δ was meant to separate the smooth and rough solutions to $D\psi = 0$. The separation is going to be reflected in the spectral flow of $H_w(m)$. To illustrate the idea, we consider the case where there is no gauge field. In the continuum, that corresponds to $A = 0$, the functions are periodic in both directions and in the discrete setting $U = 1$. Recall that the equation $D\psi = 0$ has only the constant solutions in the continuum (Theorem 2.3). However, as we know, the discretized equation $D\psi = 0$ will produce rough solutions. These solutions can be computed explicitly. First we have the following observation

Proposition 3.9. *The following collection of functions on the lattice*

$$\left\{e^{ipx}e_i \colon p = (2\pi k_1, 2\pi k_2),\ 0 \le k_1, k_2 < N,\ x \in L,\ i = 1, 2\right\}$$

form an orthonormal basis for V. (Here $px = p_1x_1 + p_2x_2$.)

Proof. We first notice that

$$e^{ip(x+e_\mu)} = e^{ipx+ip_\mu} = e^{ipx}e^{ip_\mu} = e^{ipx}$$

since $p_\mu = 2\pi k_\mu$ for some integer k_μ. Hence these functions are periodic. If they are all distinct, then there are $2N^2$ functions, which is precisely the dimension of V. To show that they form an orthonormal basis, it suffices to show that

$$\left\langle e^{ipx}, e^{iqx} \right\rangle = \delta_{pq} = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

By the definition of the inner product,

$$\left\langle e^{ipx}, e^{ipx} \right\rangle = a^2 \sum_{x\in\Gamma} e^{-ipx} e^{ipx} = a^2 N^2 = 1.$$

When $p \neq q$, put $(k_1, k_2) = q - p \neq (0,0)$ we have

$$\begin{aligned} \left\langle e^{ipx}, e^{iqx} \right\rangle &= a^2 \sum_{x\in\Gamma} e^{-ipx} e^{iqx} = a^2 \sum_{x\in\Gamma} e^{i(q-p)x} \\ &= a^2 \sum_{x\in\Gamma} e^{i(2\pi a k_1 n_1 + 2\pi a k_2 n_2)} \quad (\text{here } n_i = x_i/a) \\ &= a^2 \sum_{n_1=0}^{N-1} \left(e^{i2\pi a k_1}\right)^{n_1} \sum_{n_2=0}^{N-1} \left(e^{i2\pi a k_2}\right)^{n_2}. \end{aligned}$$

Now suppose $k_1 \neq 0$, then

$$\sum_{n_1=0}^{N-1} \left(e^{i2\pi a k_1}\right)^{n_1} = \left(\frac{1 - \left(e^{i2\pi a k_1}\right)^N}{1 - e^{i2\pi a k_1}} \right) = 0,$$

where the last equality follows because $\left(e^{i2\pi a k_1}\right)^N = e^{i2\pi a k_1 N} = 1$. □

To find solutions to $D\psi = 0$, recall that $D\psi = 0$ if and only if $D^*D = 0$ and

$$D^*D = \sum_\mu \gamma_\mu \nabla^*_\mu \sum_\nu \gamma_\nu \nabla_\nu = -\sum_\mu \gamma_\mu \nabla_\mu \sum_\nu \gamma_\nu \nabla_\nu = -\sum_\mu \nabla^2_\mu.$$

In the case where $U = 1$ we have

$$\nabla_\mu e^{ipx} = \frac{1}{2a} \left(e^{ip(x+ae_\mu)} - e^{ip(x-ae_\mu)} \right) = \frac{i}{a} \sin(ap_\mu) e^{ipx}.$$

Thus we obtain a complete list of eigenvalues of ∇_μ. It follows that

$$D^*De^{ipx} = \frac{1}{a^2} \sum_\mu \sin^2(ap_\mu) e^{ipx}.$$

(Note that here we omit the unit vector e_i for simplicity.) Hence $D^*De^{ipx} = 0$ if and only if $\sin(ap_\mu) = 0$ for all μ. The latter equations imply that $p_\mu \in \{0, \pi/a\}$ (recall that $0 \leq k_\mu < N$). Therefore

$$p \in \{(0,0), (\pi/a, 0), (0, \pi/a), (\pi/a, \pi/a)\}$$

and the kernel of D is generated by

$$\left\{1, e^{i\frac{\pi}{a}x_1}, e^{i\frac{\pi}{a}x_2}, e^{i\frac{\pi}{a}(x_1+x_2)}\right\}$$

(with the unit vectors e_i). Note that apart from 1, which gives the constant solutions in the continuum, the remaining solutions are of the form $e^{i\pi k} = (-1)^k$, which don't even converge when $a \to 0$ and these are the rough solutions.

The functions e^{ipx} are also eigenvectors of $H_w(m)^2$. Indeed,

$$H_w(m)^2 = \gamma_3(D_w - m)\gamma_3(D_w - m) = D_w^* D_w - m(D_w + D_w^*) + m^2,$$

where we've used the relation $\gamma_3 D_w = D_w^* \gamma_3$. Now in the case of no gauge field, $[D, \Delta] = 0$. Hence

$$D^*D = \left(D^* + \frac{a}{2}\Delta^*\right)\left(D + \frac{a}{2}\Delta\right) = -\sum_\mu \nabla_\mu^2 + \frac{a^2}{4}\Delta^2$$

and Δ_μ acts diagonally on e^{ipx}:

$$\Delta_\mu e^{ipx} = \frac{1}{a^2}\left(-e^{ip(x+ae_\mu)} - e^{ip(x-ae_\mu)} + 2e^{ipx}\right) = \frac{2}{a^2}(1 - \cos(ap_\mu))e^{ipx}.$$

Evaluating $H_w(m)^2$ on e^{ipx} we obtain the complete list of eigenvalues of $H_w(m)^2$:

$$H_w(m)^2 e^{ipx} = \left(\left[m - \frac{1}{a}\sum_\mu (1 - \cos(ap_\mu)\right]^2 + \frac{1}{a^2}\sum_\mu \sin^2(ap_\mu)\right) e^{ipx}.$$

Therefore the eigenvalues of $H_w(m)^2$ cross the x-axis only if $\sin(ap_\mu) = 0$ for all μ, i.e.,

$$p \in \{(0,0), (\pi/a, 0), (0, \pi/a), (\pi/a, \pi/a)\}.$$

We summarize the information in the following table:

p	$(0,0)$	$(\pi/a, 0)$	$(0, \pi/a)$	$(\pi/a, \pi/a)$
eigenvectors	1	$\exp\left(i\frac{\pi}{a}x_1\right)$	$\exp\left(i\frac{\pi}{a}x_2\right)$	$\exp\left(i\frac{\pi}{a}(x_1+x_2)\right)$
eigenvalues	m^2	$(m - 2/a)^2$	$(m - 2/a)^2$	$(m - 4/a)^2$

We see that the possible eigenvalues of $H_w(m)$ that cross the x-axis are $\pm m$, $\pm(m - a/2)$ and $\pm(m - 4/a)$ and the crossings occur at 0, $2/a$ and $4/a$ respectively. Also from the table we see that the constant solutions correspond to eigenvalue crossings at 0 and the rough solutions correspond to eigenvalue crossings far away from 0. This is what we mean by saying that Δ separates the smooth and rough solutions.

Now when the gauge field is no longer trivial, the spectral flow of $H_w(m)$ will be distorted. Nevertheless, it's been found that when the lattice gauge fields do not differ too much from the trivial case, in other words, when the lattice gauge fields satisfy certain "smoothness condition", the spectral flow of $H_w(m)$ retains certain useful features of the trivial case. In particular, the eigenvalue crossings of $H_w(m)$ occur close to 0, $2/a$ and $4/a$. Since crossings at 0 correspond to smooth solutions in the case of no gauge field, that suggests to us how we should define the index of D in the discrete setting: by the spectral flow of $H_w(m)$ in a neighborhood of 0. We'll make this idea more precise in the next section.

3.8 Approximate Smoothness Condition

Given a lattice gauge field, the quantity

$$U(x,\mu,\nu) = U_\mu(x)U_\nu(x+ae_\mu)U_\mu(x+ae_\nu)^{-1}U_\nu(x)^{-1}, \quad 1 \le \mu \ne \nu \le 2 \tag{3.5}$$

is called a *plaquette variable*. Note that this is just an oriented product of the link vari-

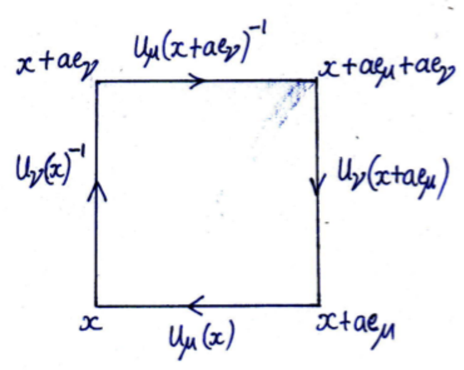

Figure 3.2: A plaquette

ables along a 2-cell in the lattice:

$$p(x,\mu,\nu) = \left\{y \in \mathbb{R}^2 \colon 0 \le y_\mu - x_\mu \le a,\ 0 \le y_\nu - x_\nu \le a\right\}, \quad 1 \le \mu \ne \nu \le 2,$$

hence the name ‘plaquette’ (Figure 3.2) . It can be checked easily that

$$U(x,\mu,\nu) = U(x,\nu,\mu)^{-1}.$$

Our main goal of this section is to establish the following result:

Theorem 3.1. *There exists a positive real number ε such that for all lattice gauge fields satisfying*

$$\max_{x\in L} |1 - U(x,\mu,\nu)| < \varepsilon, \tag{3.6}$$

the operator $H_w = H_w\left(\frac{1}{a}\right)$ has no zero eigenvalue.

Note that since $U_\mu(x+e_1) = U_\mu(x)$ for $\mu = 1, 2$ (recall condition (3.4)), we have

$$U(x+e_1,\mu,\nu) = U(x,\mu,\nu).$$

For the e_2-direction,

$$\begin{aligned} U(x+e_2,1,2) &= U_1(x+e_2)U_2(x+ae_1+e_2)U_1(x+ae_2+e_2)^{-1}U_2(x+e_2)^{-1} \\ &= e^{i\theta(x_1)}U_1(x)e^{-i\theta(x_1+a)}U_2(x+ae_1)e^{-i\theta(x_1)}U_1(x+ae_2)^{-1}e^{i\theta(x_1+a)}U_2(x)^{-1} \\ &= U(x,1,2). \end{aligned}$$

Hence

$$U(x + e_k, \mu, \nu) = U(x, \mu, \nu), \quad k = 1, 2.$$

It follows that we can restrict our attention to these lattice sites $x \in \Gamma$ in (3.6). Condition (3.6) is also called the *approximate smoothness condition*. Since the plaquette variables are unchanged under gauge transformations, gauge transformations don't change the smoothness condition.

The proof is unfortunately long and we need to make some preparations. We introduce the following operators on V

$$\begin{cases} T_\mu^+ \psi(x) = U_\mu(x)\psi(x + ae_\mu), \\ T_\mu^- \psi(x) = U_\mu(x - ae_\mu)^{-1}\psi(x - ae_\mu). \end{cases}$$

We have

$$T_\mu^+(T_\mu^-\psi)(x) = U_\mu(x)T_\mu^-\psi(x + ae_\mu) = U_\mu(x)U_\mu(x)^{-1}\psi(x) = \psi(x)$$

and similarly,

$$T_\mu^-(T_\mu^+\psi)(x) = U_\mu(x - ae_\mu)^{-1}T_\mu^+\psi(x - ae_\mu) = U_\mu(x - ae_\mu)^{-1}U_\mu(x - ae_\mu)\psi(x) = \psi(x).$$

Therefore,

$$T_\mu^+ T_\mu^- = T_\mu^- T_\mu^+ = I.$$

One nice property of these operators is that they all have unit norm. Indeed, choose any $\psi \in V$ such that $\|\psi\| = 1$, then

$$\begin{aligned} \|T_\mu^-\psi\|^2 &= \langle T_\mu^-\psi, T_\mu^-\psi \rangle \\ &= a^2 \sum_{x\in\Gamma} [T_\mu^-\psi(x)]^* T_\mu^-\psi(x) \\ &= a^2 \sum_{x\in\Gamma} U_\mu(x - ae_\mu)\psi(x - ae_\mu)^* U_\mu(x - ae_\mu)^{-1}\psi(x - ae_\mu) \\ &= \|\psi\|^2 = 1, \end{aligned}$$

where we've used the fact that $\overline{U_\mu(x - ae_\mu)} = U_\mu(x - ae_\mu)^{-1}$ in U(1). It can be shown similarly that $\|T_\mu^+\| = 1$. We are going to need the commutators of $T_\mu^\pm$, so we compute them here:

$$\begin{aligned} [T_\mu^+, T_\nu^-]\psi(x) &= T_\mu^+(T_\nu^-\psi)(x) - T_\nu^-(T_\mu^+\psi)(x) \\ &= U_\mu(x)T_\nu^-\psi(x + ae_\mu) - U_\nu(x - ae_\nu)^{-1}T_\mu^+\psi(x - ae_\nu) \\ &= U_\mu(x)U_\nu(x + ae_\mu - ae_\nu)^{-1}\psi(x + ae_\mu - ae_\nu) \\ &\quad - U_\nu(x - ae_\nu)^{-1}U_\mu(x - ae_\nu)\psi(x - ae_\nu + ae_\mu) \\ &= (U_\mu(x)U_\nu(x + ae_\mu - ae_\nu)^{-1} - U_\nu(x - ae_\nu)^{-1}U_\mu(x - ae_\nu))\psi(x - ae_\nu + ae_\mu) \\ &= U_\nu(x - ae_\nu)^{-1}(U(x - ae_\nu, \nu, \mu) - 1)U_\mu(x - ae_\nu)\psi(x - ae_\nu + ae_\mu). \end{aligned}$$

Similarly, it can be verified that

$$[T_\mu^+, T_\nu^+]\psi(x) = (U(x,\mu,\nu) - 1)U_\nu(x)U_\mu(x + ae_\nu)\psi(x + ae_\nu + ae_\mu)$$

and

$$[T_\mu^-, T_\nu^-]\psi(x) = U_\nu(x-ae_\nu)^{-1}U_\mu(x-ae_\mu-ae_\nu)^{-1}(U(x-ae_\nu-ae_\mu,\mu,\nu)-1)\psi(x-ae_\mu-ae_\nu).$$

Note that in terms of the left, right difference operators:

$$\nabla_\mu^+ = \frac{1}{a}(T_\mu^+ - I), \quad \nabla_\mu^- = \frac{1}{a}(I - T_\mu^-).$$

Hence

$$\nabla_\mu = \frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-) = \frac{1}{2a}(T_\mu^+ - T_\mu^-),$$

and

$$\Delta_\mu = \frac{1}{a}(\nabla_\mu^- - \nabla_\mu^+) = \frac{1}{a^2}(-T_\mu^+ - T_\mu^- + 2I).$$

Written in terms of $T_\mu^\pm$, we obtain several nice relations between ∇_μ and Δ_μ.

Proposition 3.10.

$$-\nabla_\mu^2 + \frac{a^2}{4}\Delta_\mu^2 - \Delta_\mu = 0.$$

Proof. Put

$$S_\mu = \frac{1}{2i}(T_\mu^+ - T_\mu^-), \quad C_\mu = \frac{1}{2}(T_\mu^+ + T_\mu^-).$$

Then

$$S_\mu^2 + C_\mu^2 = I,$$

where we've used $T_\mu^+T_\mu^- = T_\mu^-T_\mu^+ = I$. (Note the similarity with sine and cosine.) Now,

$$\nabla_\mu = \frac{1}{2a}(T_\mu^+ - T_\mu^-) = \frac{i}{a}S_\mu,$$

and

$$\Delta_\mu = \frac{1}{a^2}(-T_\mu^+ - T_\mu^- + 2I) = -\frac{2}{a^2}C_\mu + \frac{2}{a^2}.$$

Plug everything in the left hand side of the equality we get 0. □

Proposition 3.11.

$$[\nabla_\mu, \Delta_\mu] = 0.$$

Proof. We have

$$[\nabla_\mu, \Delta_\mu] = \left[\frac{1}{2a}(T_\mu^+ - T_\mu^-), \frac{1}{a^2}(-T_\mu^+ - T_\mu^- + 2I)\right].$$

Now it's clear that:

$$[T_\mu^+, I] = [T_\mu^-, I] = 0,$$

and

$$[T_\mu^+, T_\mu^-] = T_\mu^+ T_\mu^- - T_\mu^- T_\mu^+ = I - I = 0.$$

□

We're now ready to prove our theorem. It suffices to show that H_w^2 has no zero eigenvalue, i.e.,

$$\|H_w^2 \psi\| > 0$$

for all $\psi \in V$ such that $\|\psi\| = 1$. First we compute H_w^2:

$$\begin{aligned}
H_w^2 &= \gamma_3 \left(D_w - \frac{1}{a}\right) \gamma_3 \left(D_w - \frac{1}{a}\right) \\
&= \left(D_w^* - \frac{1}{a}\right)\left(D_w - \frac{1}{a}\right) \\
&= \left(-D + \frac{a}{2}\Delta - \frac{1}{a}\right)\left(D + \frac{a}{2}\Delta - \frac{1}{a}\right) \\
&= -D^2 + \frac{a}{2}(-D\Delta + \Delta D) + \left(\frac{a}{2}\Delta - \frac{1}{a}\right)^2 \\
&= -D^2 - \frac{a}{2}[D, \Delta] + \left(\frac{a}{2}\Delta - \frac{1}{a}\right)^2,
\end{aligned}$$

where in the second equality we've used $\gamma_3 D_w = -D_w^* \gamma_3$ and $\gamma_3^2 = 1$. Now we evaluate each term of the right hand side. We have:

$$\begin{aligned}
D^2 &= \sum_\mu \gamma_\mu \nabla_\mu \sum_\nu \gamma_\nu \nabla_\nu \\
&= \sum_\mu \nabla_\mu^2 + \sum_{\mu \neq \nu} \gamma_\mu \gamma_\nu \nabla_\mu \nabla_\nu \quad (\text{since } \gamma_\mu^2 = 1) \\
&= \sum_\mu \nabla_\mu^2 + \sum_{\mu > \nu} \gamma_\mu \gamma_\nu [\nabla_\mu, \nabla_\nu] \quad (\text{since } \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu),
\end{aligned}$$

and

$$\begin{aligned}
[D, \Delta] &= \left[\sum_\mu \gamma_\mu \nabla_\mu, \Delta\right] = \sum_\mu \gamma_\mu [\nabla_\mu, \Delta] \\
&= \sum_\mu \left[\nabla_\mu, \sum_\nu \Delta_\nu\right] = \sum_{\mu \neq \nu} \gamma_\mu [\nabla_\mu, \Delta_\nu],
\end{aligned}$$

where we've used $[\nabla_\mu, \Delta_\mu] = 0$ in the last equality. Finally,

$$\begin{aligned}\left(\frac{a}{2}\Delta - \frac{1}{a}\right)^2 &= \frac{a^2}{4}\Delta^2 - \Delta + \frac{1}{a^2} \\ &= \frac{a^2}{4}\left(\sum_\mu \Delta_\mu\right)^2 - \sum_\mu \Delta_\mu + \frac{1}{a^2} \\ &= \frac{a^2}{4}\sum_\mu \Delta_\mu^2 + \frac{a^2}{4}\sum_{\mu\neq\nu}\Delta_\mu\Delta_\nu - \sum_\mu \Delta_\mu + \frac{1}{a^2}.\end{aligned}$$

Note that generally $\Delta_\mu\Delta_\nu \neq \Delta_\nu\Delta_\mu$. Hence the operator H_w^2 becomes

$$\begin{aligned}H_w^2 = &-\sum_\mu \nabla_\mu^2 - \sum_{\mu>\nu}\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu] - \frac{a}{2}\sum_{\mu\neq\nu}\gamma_\mu[\nabla_\mu, \Delta_\nu] \\ &+ \frac{a^2}{4}\sum_\mu \Delta_\mu^2 + \frac{a^2}{4}\sum_{\mu\neq\nu}\Delta_\mu\Delta_\nu - \sum_\mu \Delta_\mu + \frac{1}{a^2} \\ = &\sum_\mu\left(-\nabla_\mu^2 + \frac{a^2}{4}\Delta_\mu^2 - \Delta_\mu\right) - \sum_{\mu>\nu}\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu] \\ &- \frac{a}{2}\sum_{\mu\neq\nu}\gamma_\mu[\nabla_\mu, \Delta_\nu] + \frac{a^2}{4}\sum_{\mu\neq\nu}\Delta_\mu\Delta_\nu + \frac{1}{a^2} \\ = &-\sum_{\mu>\nu}\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu] - \frac{a}{2}\sum_{\mu\neq\nu}\gamma_\mu[\nabla_\mu, \Delta_\nu] + \frac{a^2}{4}\sum_{\mu\neq\nu}\Delta_\mu\Delta_\nu + \frac{1}{a^2},\end{aligned}$$

where in the last equality we've used

$$-\nabla_\mu^2 + \frac{a^2}{4}\Delta_\mu^2 - \Delta_\mu = 0.$$

Now we can write $\Delta_\mu\Delta_\nu$ as

$$\begin{aligned}\Delta_\mu\Delta_\nu &= \left(\nabla_\mu^+\right)^* \nabla_\mu^+\Delta_\nu \\ &= \left(\nabla_\mu^+\right)^* [\nabla_\mu^+\Delta_\nu - \Delta_\nu\nabla_\mu^+ + \Delta_\nu\nabla_\mu^+] \\ &= \left(\nabla_\mu^+\right)^* [\nabla_\mu^+, \Delta_\nu] + \left(\nabla_\mu^+\right)^* \Delta_\nu\nabla_\mu^+.\end{aligned}$$

Put $P_{\mu\nu} = \left(\nabla_\mu^+\right)^* \Delta_\nu\nabla_\mu^+$, then $P_{\mu\nu}$ is positive since

$$\langle P_{\mu\nu}\psi, \psi\rangle = \left\langle \left(\nabla_\mu^+\right)^* \Delta_\nu\nabla_\mu^+\psi, \psi\right\rangle = \left\langle \Delta_\nu(\nabla_\mu^+\psi), \nabla_\mu^+\psi\right\rangle \geq 0$$

because Δ_ν is positive. Thus we can write H_w^2 as

$$H_w^2 = \frac{1}{a^2} + P - \sum_{\mu>\nu}\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu] - \frac{a}{2}\sum_{\mu\neq\nu}\gamma_\mu[\nabla_\mu, \Delta_\nu] + \frac{a^2}{4}\sum_{\mu\neq\nu}\left(\nabla_\mu^+\right)^* [\nabla_\mu^+, \Delta_\nu]$$

where P is a positive operator. Now for any vector $\psi \in V$ with $\|\psi\| = 1$,

$$\begin{aligned}\|H_w^2\psi\| \geq \left\| \left(\frac{1}{a^2} + P\right)\psi \right\| - \sum_{\mu>\nu} \|\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu]\psi\| - \frac{a}{2}\sum_{\mu\neq\nu} \|\gamma_\mu[\nabla_\mu, \Delta_\nu]\psi\| \\ - \frac{a^2}{4}\sum_{\mu\neq\nu} \left\|(\nabla_\mu^+)^*[\nabla_\mu^+, \Delta_\nu]\psi\right\|.\end{aligned}$$

Now since P is positive,

$$\begin{aligned}\left\|\left(\frac{1}{a^2}+P\right)\psi\right\|^2 &= \left\langle \left(\frac{1}{a^2}+P\right)\psi, \left(\frac{1}{a^2}+P\right)\psi \right\rangle \\ &= \left\langle \frac{1}{a^2}\psi, \frac{1}{a^2}\psi \right\rangle + \left\langle \frac{1}{a^2}\psi, P\psi \right\rangle + \left\langle P\psi, \frac{1}{a^2}\psi \right\rangle + \langle P\psi, P\psi\rangle \\ &\geq \left\|\frac{1}{a^2}\psi\right\|^2 = \frac{1}{a^2}.\end{aligned}$$

Also we have the following (recall that $\|\psi\| = 1$ and $\|\gamma_\mu\| = 1$):

$$\|\gamma_\mu\gamma_\nu[\nabla_\mu, \nabla_\nu]\psi\| \leq \|\gamma_\mu\|\|\gamma_\nu\| \, \|[\nabla_\mu, \nabla_\nu]\| \, \|\psi\| = \|[\nabla_\mu, \nabla_\nu]\|,$$

and

$$\|\gamma_\mu[\nabla_\mu, \Delta_\nu]\psi\| \leq \|\gamma_\mu\| \, \|[\nabla_\mu, \Delta_\nu]\| \, \|\psi\| = \|\nabla_\mu, \Delta_\nu\|,$$

and

$$\left\|(\nabla_\mu^+)^*[\nabla_\mu^+, \Delta_\nu]\psi\right\| \leq \left\|(\nabla_\mu^+)^*\right\| \, \|[\nabla_\mu^+, \Delta_\nu]\|\|\psi\| \leq \frac{2}{a}\|[\nabla_\mu^+, \Delta_\nu]\|,$$

where

$$\|(\nabla_\mu^+)^*\| = \|\nabla_\mu^-\| = \left\|\frac{1}{a}(I - T_\mu^-)\right\| \leq \frac{1}{a}(1 + \|T_\mu^-\|) = \frac{2}{a}.$$

(Recall that $\|T_\mu^\pm\| = 1$.) Consequently,

$$\|H_w^2\psi\| \geq \frac{1}{a^2} - \sum_{\mu>\nu} \|[\nabla_\mu, \nabla_\nu]\| - \frac{a}{2}\sum_{\mu\neq\nu} \|[\nabla_\mu, \Delta_\nu]\| - \frac{a}{2}\sum_{\mu\neq\nu} \|[\nabla_\mu^+, \Delta_\nu]\|.$$

Now note that

$$\begin{aligned}[\nabla_\mu^+, \nabla_\mu^-] &= \left[\frac{1}{a}(T_\mu^+ - I), \frac{1}{a}(I - T_\nu^-)\right] = -\frac{1}{a^2}[T_\mu^+, T_\nu^-], \\ [\nabla_\mu^+, \nabla_\nu^+] &= \left[\frac{1}{a}(T_\mu^+ - I), \frac{1}{a}(T_\nu^+ - I)\right] = \frac{1}{a^2}[T_\mu^+, T_\nu^+], \\ [\nabla_\mu^-, \nabla_\nu^-] &= \left[\frac{1}{a}(I - T_\mu^-), \frac{1}{a}(I - T_\nu^-)\right] = \frac{1}{a^2}[T_\mu^-, T_\nu^-].\end{aligned}$$

Thus we need to investigate the norms of the commutators of $T_\mu^\pm$. Put $R = [T_\mu^+, T_\nu^-]$. Then for any vector $\psi \in V$ such that $\|\psi\| = 1$, we have

$$\begin{aligned}\|R\psi\|^2 = \langle R\psi, R\psi\rangle &= a^2 \sum_{x\in\Gamma} R\psi(x)^* R\psi(x) \\ &= a^2 \sum_{x\in\Gamma} |U(x - ae_\nu, \nu, \mu) - 1|^2 \psi(x)^*\psi(x) \\ &< \max_{x\in\Gamma} |U(x - ae_\nu, \nu, \mu) - 1|^2 = \varepsilon^2.\end{aligned}$$

Thus $\|[T_\mu^+, T_\nu^-]\| < \varepsilon$. Similarly we can show that

$$\left\|[T_\mu^+, T_\nu^+]\right\| < \varepsilon, \quad \left\|[T_\mu^-, T_\nu^-]\right\| < \varepsilon.$$

Now we can bound the commutators:

$$\|[\nabla_\mu, \nabla_\nu]\| = \left\|\left[\frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-), \frac{1}{2}(\nabla_\mu^+ + \nabla_\nu^-)\right]\right\| \le \frac{1}{4}\left(4 \times \frac{1}{a^2}\varepsilon\right) = \frac{\varepsilon}{a^2},$$

and

$$\|[\nabla_\mu, \Delta_\nu]\| = \left\|\left[\frac{1}{2}(\nabla_\mu^+ + \nabla_\mu^-), \frac{1}{a}(\nabla_\nu^- - \nabla_\nu^+)\right]\right\| \le \frac{1}{2a}\left(4 \times \frac{1}{a^2}\varepsilon\right) = \frac{2\varepsilon}{a^3},$$

and

$$\left\|[\nabla_\mu^+, \Delta_\nu]\right\| = \left\|\left[\nabla_\mu^+, \frac{1}{a}(\nabla_\nu^- - \nabla_\nu^+)\right]\right\| \le \frac{1}{a}\left(2 \times \frac{1}{a^2}\varepsilon\right) = \frac{2\varepsilon}{a^3}.$$

Hence

$$\begin{aligned}\|H_w^2\psi\| &> \frac{1}{a^2} - \frac{\varepsilon}{a^2} - \frac{a}{2} \times 2 \times \frac{2\varepsilon}{a^3} - \frac{a}{2} \times 2 \times \frac{2\varepsilon}{a^3} \\ &= \frac{1}{a^2}(1 - 5\varepsilon).\end{aligned}$$

So if $0 < \varepsilon < 1/5$, then $\|H_w^2\psi\| > 0$ or in other words, H_w has no zero eigenvalue and our proof is complete. □

One important consequence of Theorem 3.1 is

Corollary 3.1. *There are no eigenvalue crossings of the x-axis in the interval*

$$\left(\frac{1}{a} - \frac{1}{a}\sqrt{1 - 5\varepsilon}, \frac{1}{a} + \frac{1}{a}\sqrt{1 - 5\varepsilon}\right)$$

for $0 < \varepsilon < 1/5$.

Proof. Put $b = 1/a\sqrt{1 - 5\varepsilon}$. Suppose $\lambda(m)$ is an eigenvalue of $H_w(m)$. From Theorem 3.1 we have for some unit vector $\psi \in V$,

$$\left|\lambda\left(\frac{1}{a}\right)\right| = \|H_w\psi\| > \frac{1}{a}\sqrt{1 - 5\varepsilon} = b.$$

Now suppose there's an eigenvalue crossing of the x-axis in the interval $(1/a - b, 1/a + b)$, i.e.,

$$\lambda(x_0) = 0$$

for some $x_0 \in (1/a - b, 1/a + b)$. By the Mean Value Theorem, there exists ξ between x_0 and $1/a$ such that

$$|\lambda'(\xi)| = \left| \frac{\lambda(1/a) - \lambda(x_0)}{1/a - x_0} \right| = \left| \frac{\lambda(1/a)}{1/a - x_0} \right|.$$

Since

$$|\lambda(1/a)| > b > |1/a - x_0|,$$

we obtain $|\lambda'(\xi)| > 1$, contradicting the fact that $|\lambda'(x)| \leq 1$ (Proposition 3.8). $\square$

Thus it makes sense to talk about eigenvalue crossings close to 0 (specifically eigenvalue crossings in the interval $(0, 1/a)$). Generally, it's proved in [4] that the eigenvalue crossings of $H_w(m)$ only occur close to 0, 2 and 4 in unit of $1/a$ (here we are working with 2 dimensional lattice, in the case of d dimensional lattice the crossings occur close to 0,2,...,2d). We define the index of D to be minus the spectral flow of $H_w(m)$ in the interval $(0, 1/a)$.

We'd like to find an explicit formula for the spectral flow of $H_w(m)$. Since $H_w(m)$ is hermitian, the operator $H_w(m)^2$ is positive and hence it has the positive square root $\sqrt{H_w(m)^2}$ (Subsection 1.2.5). The eigenvalues of $\sqrt{H_w(m)^2}$ are $|\lambda_i|$, where λ_i are eigenvalues of $H_w(m)$. In the case where $H_w(m)$ has no zero eigenvalue, the operator $\sqrt{H_w(m)^2}$ is invertible and the eigenvalues of $H_w(m)/\sqrt{H_w(m)^2}$ are

$$\frac{\lambda_i}{|\lambda_i|} = \begin{cases} +1 & \text{if } \lambda_i > 0 \\ -1 & \text{if } \lambda_i < 0 \end{cases} = \text{sign}\,(\lambda_i).$$

Therefore the following quantity

$$\text{tr}\left(\frac{H_w(m)}{\sqrt{H_w(m)^2}} \right)$$

equals the difference between positive eigenvalues and negative eigenvalues of $H_w(m)$. Recall that eigenvalues of $H_w(m)$ only cross the x-axis when $m \geq 0$. Therefore when $m < 0$ $H_w(m)$ has no zero eigenvalues and the difference between positive eigenvalues and negative eigenvalues of $H_w(m)$ is constant. Even more is true:

Proposition 3.12. *$H_w(m)$ has the same number of positive and negative eigenvalues when $m < 0$.*

Proof. From the above discussion we have that the quantity

$$\text{tr}\left(\frac{H_w(m)}{\sqrt{H_w(m)^2}} \right)$$

is constant when $m < 0$. We have

$$\begin{aligned}
\lim_{m\to-\infty} \operatorname{tr}\left(\frac{H_w(m)}{\sqrt{H_w(m)^2}}\right) &= \lim_{m\to-\infty} \operatorname{tr}\left(\frac{\frac{1}{|m|}H_w(m)}{\sqrt{\frac{1}{m^2}H_w(m)^2}}\right) \\
&= \lim_{m\to-\infty} \operatorname{tr}\left(\frac{\frac{1}{|m|}\gamma_3 D_w - \frac{m}{|m|}\gamma_3}{\sqrt{\frac{1}{m^2}D_w^* D_w - \frac{1}{m}(D_w^* + D_w) + 1}}\right) \\
&= \operatorname{tr}(\gamma_3 \colon V \to V) \\
&= 0.
\end{aligned}$$

Thus $H_w(m)$ has the same number of positive and negative eigenvalues when $m < 0$. $\square$

We now come to our important result:

Proposition 3.13. *The index of D is given by*

$$\operatorname{Index}(D) = -\frac{1}{2}\operatorname{tr}\left(\frac{H_w}{\sqrt{H_w^2}}\right),$$

where $H_w = H_w(1/a)$.

Proof. Note that the right hand side is well-defined since we proved that H_w has no zero eigenvalue. Recall that $\operatorname{Index}(D)$ is minus the spectral flow of $H_w(m)$ in the interval $(0, 1/a)$. Put

$$\eta(m) = \#\text{+ve eigenvalues of } H_w(m) - \#\text{−ve eigenvalues of } H_w(m)$$

when m is far away from $0, 2/a, 4/a$. We know that $\eta(m) = 0$ when $m < 0$. When $m \geq 0$, every time there's an eigenvalue crossing with positive slope, $\eta(m)$ will increase by 2 and it will decrease by 2 every time there's an eigenvalue crossing with negative slope. Thus we conclude that $\eta(1/a)$ is twice the difference between eigenvalue crossings with positive slope and eigenvalue crossings with negative slope in $(0, 1/a)$. In other words, the spectral flow of $H_w(m)$ in $(0, 1/a)$ is precisely

$$\frac{1}{2}\eta(1/a) = \frac{1}{2}\operatorname{tr}\left(\frac{H_w}{\sqrt{H_w^2}}\right)$$

and that completes the proof. $\square$

By defining the (discrete) index of D in term of the spectral flow of $H_w(m)$, we hope to obtain some meaningful information about the index of D in the continuum. Specifically, we expect that when the lattice gauge field is the transcript of some smooth gauge field, the (discrete) index of D will reproduce the correct index when we take $a \to 0$. Proving this rigorously is not so trivial and a detailed proof can be found in [2].

We remark that under a gauge transformation $e^{i\phi}$, a basis β for V consisting of eigenvectors of H_w will be transformed into $e^{i\phi}\beta$, which can be verified easily to be a basis for $\widetilde{V}$. Moreover, if $H_w f_i = \lambda_i f_i$, then

$$\widetilde{H}_w(e^{i\phi} f_i) = e^{i\phi} H_w f = \lambda_i e^{i\phi} f_i$$

by properties of gauge transformations. Thus H_w and $\widetilde{H}_w$ have the same eigenvalues counting multiplicities. It follows that the index of D is gauge invariant.

3.9 Discrete Topological Charge

In this section we aim to give a meaningful notion of the topological charge associated to a lattice gauge field. Recall that for lattice gauge fields the usual definition of the topological charge based on the twisted periodicity condition does not work since in the discrete case

$$\theta(x_1 + 1) = \theta(x_1) + 2\pi Q(x_1)$$

and in general the integer Q depends on x_1. To facilitate our discussion, we restrict our attention to lattice gauge fields whose plaquette variables (3.5) are different from -1 and we consider the principal branch of the logarithmic function:

$$\log e^{i\phi} = i\phi,$$

where $\phi \in (-\pi, \pi)$. The next proposition plays an important role in our definition of the discrete topological charge.

Proposition 3.14. *The following quantity*

$$\frac{1}{2i\pi} \sum_{x\in\Gamma,\mu<\nu} \log U(x,\mu,\nu)$$

is an integer.

Proof. The key observation is the following identity

$$\prod_{x\in\Gamma,\mu<\nu} U(x,\mu,\nu) = 1.$$

This identity is best explained visually. When we multiply the plaquette variables, because of the orientations that we chose for the plaquettes as well as the fact that the product of two link variables with opposite orientations is one, we end up with product of the link variables along the boundary of the square $[0,1]^2$. Now using the twisted periodicity condition (3.4) we obtain

$$\begin{aligned}\prod_{x\in\Gamma,\mu<\nu} U(x,\mu,\nu) &= e^{i(\theta(x_1+a)-\theta(x_1))} e^{i(\theta(x_1+2a)-\theta(x_1+a))} \cdots e^{i(\theta(x_1+1)-\theta(x_1+(N-1)a))} \\ &= e^{i(\theta(x_1+1)-\theta(x_1))} = e^{i2\pi Q(x_1)} \\ &= 1.\end{aligned}$$

It follows from the properties of the exponential function that

$$\begin{aligned}
\exp\left(\sum_{x\in\Gamma,\mu<\nu}\log U(x,\mu,\nu)\right) &= \prod_{x\in\Gamma,\mu<\nu}\exp\left(\log U(x,\mu,\nu)\right)\\
&= \prod_{x\in\Gamma,\mu<\nu} U(x,\mu,\nu)\\
&= 1.
\end{aligned}$$

Thus

$$\sum_{x\in\Gamma,\mu<\nu}\log U(x,\mu,\nu) = i2\pi Q$$

for some integer Q. □

Note that although we only consider $x\in\Gamma$, $U(x,\mu,\nu)$ is unchanged when we go in a distance of one in any direction. Thus the integer Q that we obtain is independent of the lattice sites and we define it to be our topological charge. Thus the *topological charge* of a lattice gauge field is given by

$$Q = \frac{1}{i2\pi}\sum_{x\in\Gamma,\mu<\nu}\log U(x,\mu,\nu).$$

Of course we assume that $U(x,\mu,\nu)\neq -1$ (so that its angle lies in $(-\pi,\pi)$).

To justify our definition of the topological charge, we need to show that our discrete topological charge agrees with its continuum analogue when we take the limit $a\to 0$. When the lattice gauge field U is the lattice transcript of some smooth gauge field A, recall that (Section 3.3):

$$U_\mu(x) = 1 + iaA_\mu(x) + i\frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) - \frac{a^2}{2}A_\mu^2(x) + \mathcal{O}(a^3),$$

and

$$U_\mu(x)^{-1} = 1 - iaA_\mu(x) - i\frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) - \frac{a^2}{2}A_\mu^2(x) + \mathcal{O}(a^3).$$

Thus we have

$$\begin{aligned}
U_\mu(x)U_\nu(x+ae_\nu) = 1 &+ iaA_\mu(x) + iaA_\nu(x+ae_\mu) + i\frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x) + i\frac{a^2}{2}\frac{\partial A_\nu}{\partial x_\nu}(x+ae_\mu)\\
&- \frac{a^2}{2}A_\mu^2(x) - \frac{a^2}{2}A_\nu^2(x+ae_\mu) - a^2A_\mu(x)A_\nu(x+ae_\mu) + \mathcal{O}(a^3),
\end{aligned}$$

and

$$\begin{aligned}
U_\mu(x+ae_\nu)^{-1}U_\nu(x)^{-1} = 1 &- iaA_\mu(x+ae_\nu) - iaA_\nu(x) - i\frac{a^2}{2}\frac{\partial A_\mu}{\partial x_\mu}(x+ae_\nu) - i\frac{a^2}{2}\frac{\partial A_\nu}{\partial x_\nu}(x)\\
&- \frac{a^2}{2}A_\mu^2(x+ae_\nu) - \frac{a^2}{2}A_\nu^2(x) - a^2A_\mu(x+ae_\nu)A_\nu(x) + \mathcal{O}(a^3).
\end{aligned}$$

Now since A_μ are smooth, we have the following Taylor expansions:

$$A_\mu(x + ae_\nu) = A_\mu(x) + \mathcal{O}(a),$$

and

$$A_\mu^2(x + ae_\nu) = (A_\mu(x) + \mathcal{O}(a))^2 = A_\mu^2(x) + \mathcal{O}(a),$$

and

$$\frac{\partial A_\mu}{\partial x_\mu}(x + ae_\nu) = \frac{\partial A_\mu}{\partial x_\mu}(x) + \mathcal{O}(a).$$

Plug these expansions in the expressions for $U_\mu(x)U_\nu(x+ae_\mu)$ and $U_\mu(x+ae_\nu)^{-1}U_\nu(x)^{-1}$ and multiply them together we obtain the expression for the plaquette variable $U(x,\mu,\nu)$:

$$\begin{aligned} U(x,\mu,\nu) &= 1 + ia\left[(A_\mu(x) - A_\mu(x+ae_\nu)) + (A_\nu(x+ae_\mu) - A_\nu(x))\right] + \mathcal{O}(a^3) \\ &= 1 + ia^2\left(\frac{\partial A_\nu}{\partial x_\mu}(x) - \frac{\partial A_\mu}{\partial x_\nu}(x)\right) + \mathcal{O}(a^3) \\ &= 1 + ia^2F_{\mu\nu}(x) + \mathcal{O}(a^3), \end{aligned}$$

where

$$F_{\mu\nu}(x) = \frac{\partial A_\nu}{\partial x_\mu}(x) - \frac{\partial A_\mu}{\partial x_\nu}(x)$$

(recall Section 2.4). Actually these $\mathcal{O}(a^3)$ should depend on x. However, since the partial derivatives of A_μ are bounded, we can write $\mathcal{O}(a^3)$ independent of x. What it means is that for a small enough there exists some $M > 0$ such that

$$|U(x,\mu,\nu) - 1 - iaF_{\mu\nu}(x)| < Ma^3$$

for all x. To proceed we recall the following series expansion of the principal logarithm:

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z| < 1.$$

In our case we can choose a to be small enough so that

$$|1 - U(x,\mu,\nu)| < 1$$

for all x. Therefore,

$$\log U(x,\mu,\nu) = ia^2F_{\mu\nu} + \mathcal{O}(a^3).$$

Now we have

$$\begin{aligned} Q_{\text{lattice}} &= \frac{1}{i2\pi} \sum_{x\in\Gamma,\mu<\nu} \log U(x,\mu,\nu) \\ &= \frac{1}{2\pi} \sum_{x\in\Gamma,\mu<\nu} a^2F_{\mu\nu}(x) + \mathcal{O}(a^3) \\ &\xrightarrow{a\to 0} \frac{1}{2\pi} \int_{[0,1]^2} F_{12}(x)\,dx_1dx_2 \\ &= Q_{\text{continuum}}. \end{aligned}$$

Note that for a small enough the term

$$\sum_{x\in\Gamma} |\mathcal{O}(a^3)| < \sum_{x\in\Gamma} Ma^3 = MN^2a^3 = Ma,$$

which tends to 0 when a tends to 0. We've shown that the discrete topological charge reproduces the topological charge in the continuum. Note however that contrary to the continuum case, our definition of the topological charge may depend on the lattice gauge field (and so does $\mathrm{Index}\, D$). Since the plaquette variables $U(x, \mu, \nu)$ don't change under a gauge transformation, as can be checked easily, the topological charge is also gauge invariant.

3.10 The Main Problem

Having developed all the necessary ingredients, we want to establish a counterpart of the Atiyah-Singer Index Theorem on the lattice. In other words, we want to show

$$\mathrm{Index}\,(D)_{\text{lattice}} = Q_{\text{lattice}},$$

where

$$\mathrm{Index}\,(D)_{\text{lattice}} = -\frac{1}{2}\mathrm{tr}\,\frac{H_w(1/a)}{\sqrt{H_w^2(1/a)}},$$

and

$$Q_{\text{lattice}} = \frac{1}{2\pi i}\sum_{x\in\Gamma,\mu<\nu} \log U(x, \mu, \nu).$$

For the index and the topological charge to be well-defined, we require the lattice gauge field to satisfy an approximate smoothness condition:

$$\max_{x,\mu,\nu} |1 - U(x, \mu, \nu)| < \varepsilon$$

for some small positive real number ε. The difficulty lies in the fact that the lattice gauge fields are quite arbitrary, so it's not easy to write down explicitly the eigenvalues of H. One possible approach to the problem is as follows.

- Given a lattice gauge field U, we first construct a continuum gauge field A (actually piecewise smooth suffices) whose lattice transcript is U and whose topological charge Q_A is the same as that of U.

- Equipped with a continuum gauge field A, we'll try to show that the (discrete) index of D_U is unchanged under successively halving the lattice, i.e., halving the lattice spacing a. More specifically, starting with a lattice with spacing a, we construct a lattice with spacing $a/2$ on which we put the lattice gauge field U' obtained from taking the transcript of the continuum gauge field A. We need to make sure that U' still satisfies the smoothness condition so that the index of $D_{U'}$ is well-defined. We want to show that

$$\text{Index}\, D_{U'} = \text{Index}\, D_U.$$

The same proof can be applied to lattices with spacings $a/4, a/8, ..., a/2^n, ...$ It follows that

$$\text{Index}\, D_U = \lim_{a \to 0} \text{Index}\, D_{U'}.$$

We know that in the continuum limit, the index of $D_{U'}$ reproduces the topological charge of the continuum gauge field A, which is the topological charge Q_U of U in this case. Thus

$$\text{Index}\, D_U = Q_U.$$

We note that in the lattice setting, it's possible to convert the twisted periodicity conditions associated with the lattice gauge field to periodicity conditions via the gauge transformation:

$$e^{i\phi(x)} = 1 \quad \text{for } x \in \Gamma$$

and extend to the to the whole of $\mathbb{R}^2$ by the condition

$$\begin{cases} e^{i\phi(x+e_1)} = e^{i\phi(x)}, \\ e^{i\phi(x+e_2)} = e^{-i\theta(x_1)} e^{i\phi(x)}. \end{cases}$$

Our vector space becomes

$$\widetilde{V} = \left\{ e^{i\phi} f \colon f \in V \right\},$$

which consists of periodic functions since

$$e^{i\phi(x+e_1)} f(x+e_1) = e^{i\phi(x)} f(x),$$

and

$$e^{i\phi(x+e_2)} f(x+e_2) = e^{-i\theta(x_1)} e^{i\phi(x)} e^{i\theta(x_1)} f(x) = e^{i\phi(x)} f(x).$$

Our lattice gauge field will also be transformed accordingly:

$$\widetilde{U}_\mu(x) = e^{i\phi(x)} U_\mu(x) e^{-i\phi(x+ae_\mu)}$$

and it is also periodic. For $\widetilde{U}_1$ we have

$$\widetilde{U}_1(x+e_1) = e^{i\phi(x+e_1)} U_1(x+e_1) e^{-\phi(x+ae_1+e_1)} = e^{i\phi(x)} U_1(x) e^{-i\phi(x+ae_1)} = \widetilde{U}_1(x),$$

and

$$\begin{aligned} \widetilde{U}_1(x+e_2) &= e^{i\phi(x+e_2)} U_1(x+e_2) e^{-i\phi(x+ae_1+e_2)} \\ &= e^{-i\theta(x_1)} e^{i\phi(x)} e^{i\theta(x_1)} U_1(x) e^{-i\theta(x_1+a)} e^{i\theta(x_1+a)} e^{-i\phi(x+ae_1)} \\ &= e^{i\phi(x)} U_1(x) e^{-i\phi(x+ae_1)} = \widetilde{U}_1(x). \end{aligned}$$

For $\widetilde{U}_2$ we have

$$\widetilde{U}_2(x+e_1) = e^{i\phi(x+e_1)}U_2(x+e_1)e^{-i\phi(x+e_1+ae_2)} = e^{i\phi(x)}U_2(x)e^{-i\phi(x+ae_2)} = \widetilde{U}_2(x),$$

and

$$\begin{aligned}\widetilde{U}_2(x+e_2) &= e^{i\phi(x+e_2)}U_2(x+e_2)e^{-i\phi(x+e_2+ae_2)}\\ &= e^{-i\theta(x_1)}e^{i\phi(x)}U_2(x)e^{i\theta(x_1)}e^{-i\phi(x+ae_2)}\\ &= e^{i\phi(x)}U_2(x)e^{-i\phi(x+ae_2)} = \widetilde{U}_2(x).\end{aligned}$$

Since the index of D and the topological charge Q_U are gauge invariant, we can work with the periodicity conditions instead of the twisted periodicity conditions. (This is another reason why we don't define the topological charge in terms of the boundary condition, it's not gauge invariant.) One advantage of working with the periodicity condition is we can use the basis functions e^{ipx}. This trick however doesn't work in the continuum. Suppose A is has nonzero topological charge Q and $\widetilde{A}$ is periodic, then it follows that

$$\frac{1}{2\pi}\int_{[0,1]^2} \widetilde{F}_{12}(x_1,x_2)dx_1dx_2 = 0,$$

which is a contradiction since gauge transformations don't change the topological charge. (This is true because gauge transformations are smooth. In other words, if we want to change the topological charge via a gauge transformation, we have to introduce discontinuities somewhere.)

3.10.1 Piecewise Smooth Gauge Fields from Lattice Gauge Fields

Given a periodic lattice gauge field U with topological charge Q_U, which satisfies the approximate smoothness condition, we can write

$$U_\mu(x) = e^{i\alpha_\mu(x)}.$$

Note that α_μ can be defined up to an integral multiple of 2π. Here we assume that α_μ can be chosen in such a way that the sum of the angles around a plaquette lies between $-\pi$ and π, i.e.,

$$-\pi < \alpha_1(x) + \alpha_2(x+ae_1) - \alpha_1(x+ae_2) - \alpha_2(x) < \pi.$$

(The detailed description will be given below.) It suffices to construct the gauge field A for each plaquette $p(x,\mu,\nu)$. The construction is quite simple: we want the gauge field A to be constant along the boundary of the plaquette and vary continuously along a straight line inside the plaquette. Since the transcript of A is U, we have

$$\alpha_\mu(x) = \int_0^a A_\mu(x+(a-t)e_\mu)dt.$$

Hence

$$A_\mu(x + (a - t)e_\mu) = \frac{1}{a}\alpha_\mu(x), \quad t \in [0, a].$$

Similarly for $\mu \neq \nu$,

$$A_\mu(x + (a - t)e_\mu + ae_\nu) = \frac{1}{a}\alpha_\mu(x + ae_\nu), \quad t \in [0, a].$$

Inside the plaquette, A_μ is given by

$$A_\mu(x + (a - s)e_\nu + (a - t)e_\mu) = \left(\frac{a - s}{a^2}\right)\alpha_\mu(x + ae_\nu) + \frac{s}{a^2}\alpha_\mu(x), \quad 0 \leq t, s \leq a$$

for $\mu \neq \nu$. The gauge field A is smooth everywhere except for a set of jump discontinuities of measure 0. Specifically, A_1 is discontinuous along the vertical links and A_2 is discontinuous along the horizontal links. The topological charge of A is however still well-defined:

$$Q_A = \frac{1}{2\pi}\int_{[0,1]^2} F_{12}(x_1, x_2)dx_1dx_2 = \frac{1}{2\pi}\sum_{x\in\Gamma}\int_{p(x)} F_{12}(x_1, x_2)dx_1dx_2.$$

Now for each plaquette $p(x)$,

$$\begin{aligned}
\int_{p(x)} F_{12}dx_1dx_2 &= \int_{[0,a]^2} F_{12}(x + (a - t_1)e_1 + (a - t_2)e_2)dt_1dt_2, \\
&= \int_0^a \left(\int_0^a \frac{\partial A_2}{\partial x_1}(x_1 + a - t_1, x_2 + a - t_2)dt_1\right) dt_2 \\
&\qquad - \int_0^a \left(\int_0^a \frac{\partial A_1}{\partial x_2}(x_1 + a - t_1, x_2 + a - t_2)dt_2\right) dt_1 \\
&= \alpha_2(x + ae_1) - \alpha_2(x) - \alpha_1(x + ae_2) + \alpha_1(x) \\
&= \frac{1}{i}\log U(x, 1, 2),
\end{aligned}$$

where the third equality follows since A_μ is constant along the μ direction and the last equality follows from the sum of the angles is between $-\pi$ and π. Thus

$$Q_A = \frac{1}{2\pi}\sum_{x\in\Gamma}\int_{p(x)} F_{12}dx_1dx_2 = \frac{1}{2i\pi}\sum_{x\in\Gamma}\log U(x, 1, 2) = Q_U.$$

The topological charge of A is the same as that of U.

Now we investigate the effect of halving the lattice (Figure 3.3). After we halve the lattice, we can put on it a lattice gauge field obtained by taking the transcript of the gauge field A constructed above. Notice that A_μ is constant along the μ direction, thus

$$U(x, x + (a/2)e_\mu) = \exp\left(i\int_0^{a/2} A_\mu(x + (a/2 - t)e_\mu)dt\right) = e^{i\alpha_\mu(x)/2}.$$

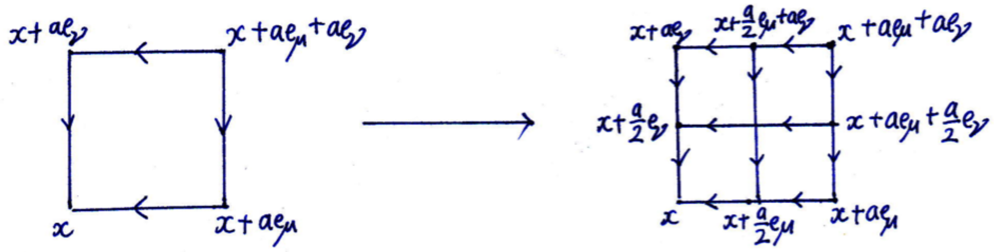

Figure 3.3: Halving the lattice

Similarly,

$$U(x + a/2e_\mu, x + ae_\mu) = e^{i\alpha_\mu(x)/2}.$$

So along each old link of the lattice, the angles are half the original ones. Along each new link,

$$A_\mu(x + (a/2)e_\nu + (a-t)e_\mu) = \frac{1}{2}\left(\frac{\alpha_\mu(x + ae_\nu)}{a} + \frac{\alpha_\mu(x)}{a}\right).$$

The lattice transcript is

$$\begin{aligned} U(x + (a/2)e_\nu, x + (a/2)e_\nu + (a/2)e_\mu) &= \exp\left(i\frac{a}{2}\cdot\frac{\alpha_\mu(x + ae_\nu) + \alpha_\mu(x)}{2a}\right) \\ &= \exp\left(i\frac{\alpha_\mu(x) + \alpha_\mu(x + ae_\nu)}{4}\right). \end{aligned}$$

So the angles along each new link are given by

$$\begin{aligned} \alpha(x + (a/2)e_\nu, x + (a/2)e_\nu + (a/2)e_\mu) &= \frac{\alpha_\mu(x) + \alpha_\mu(x + ae_\nu)}{4} \\ &= \alpha(x + (a/2)e_\nu + (a/2)e_\mu, x + (a/2)e_\nu + ae_\mu). \end{aligned}$$

Let us focus on a particular plaquette. In Figure 3.4, α_i denote the angles with orientations (so it will pick up a minus sign when the orientation is reversed) and

$$\alpha_{13} = \frac{\alpha_1 - \alpha_3}{2}, \quad \alpha_{24} = \frac{\alpha_2 - \alpha_4}{2}.$$

By assumption, we have $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is between $-\pi$ and π. We label the subplaquettes of the divided plaquette according to the figure. The sums of the angles along

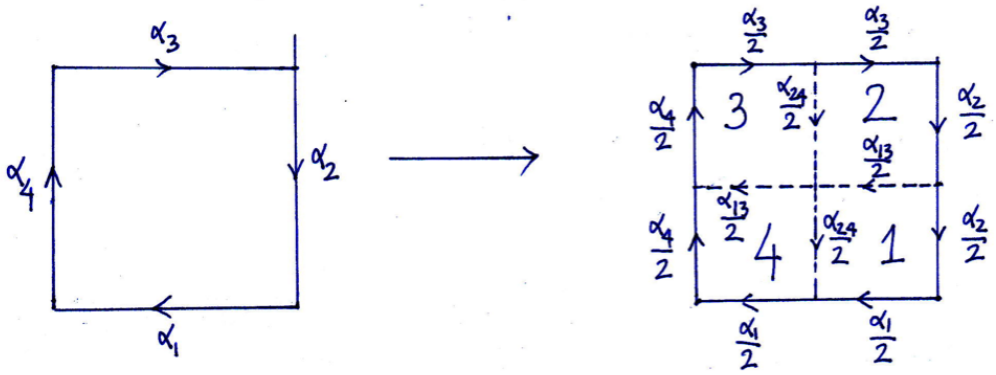

Figure 3.4: Subplaquettes and their angles

each subplaquette are computed below (in the order of the subplaquettes):

$$\begin{aligned}
\frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{\alpha_{13}}{2}-\frac{\alpha_{24}}{2} &= \frac{\alpha_1}{2}+\frac{\alpha_2}{2}-\frac{\alpha_1-\alpha_3}{4}-\frac{\alpha_2-\alpha_4}{4}=\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4},\\
\frac{\alpha_{13}}{2}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}-\frac{\alpha_{24}}{2} &= \frac{\alpha_1-\alpha_3}{4}+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}-\frac{\alpha_2-\alpha_4}{4}=\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4},\\
\frac{\alpha_{13}}{2}+\frac{\alpha_{24}}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2} &= \frac{\alpha_1-\alpha_3}{4}+\frac{\alpha_2-\alpha_4}{4}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}=\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4},\\
\frac{\alpha_1}{2}+\frac{\alpha_{24}}{2}-\frac{\alpha_{13}}{2}+\frac{\alpha_4}{2} &= \frac{\alpha_1}{2}+\frac{\alpha_2-\alpha_4}{4}-\frac{\alpha_1-\alpha_3}{4}+\frac{\alpha_4}{2}=\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}.
\end{aligned}$$

Thus the sum of the angles along each subplaquette is also between $-\pi$ and π. Also,

$$|1-e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)/4}| < |1-e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}| < \varepsilon.$$

(Note that this is only true if $-\pi < \alpha_1+\alpha_2+\alpha_3+\alpha_4 < \pi$.) Hence the smoothness condition $\max|1-U(x)| < \varepsilon$ is still satisfied. It follows that the index of D is well-defined under successively halving the lattice. The proof in [2] still applies to the case of piecewise smooth lattice gauge field and we have

$$\lim_{a\to 0} \operatorname{Index} D = Q_A.$$

Now we describe how to choose the angles α_μ in such a way that the sum of the angles around a plaquette lies between $-\pi$ and π. We first start with the plaquettes $p(x)$ for x lies on the horizontal axis. For the upward direction, let $\alpha_1(x)$, $\alpha_2(x+ae_1)$, $\alpha_2(x)$ be the unique angles in $[-\pi,\pi]$ and choose $\alpha_1(x+ae_2)$ such that

$$-\pi < \alpha_1(x)+\alpha_2(x+ae_1)-\alpha_1(x+ae_2)-\alpha_2(x) < \pi.$$

Note that $\alpha_1(x+ae_2)$ is not necessarily in $[-\pi,\pi]$. (The approximate smoothness condition ensures that the angle of a link variable lies between $-\pi$ and π up to some integral multiple of 2π and hence the choice of $\alpha_1(x + ae_2)$ is unique.) For the downward direction, let $\alpha_1(x)$, $\alpha_2(x - ae_2)$, $\alpha_2(x + ae_1 - ae_2)$ be the unique angles in $[-\pi, \pi]$ and choose $\alpha_1(x - ae_2)$ such that

$$-\pi < \alpha_1(x - ae_2) + \alpha_2(x + ae_1 - ae_2) - \alpha_1(x) - \alpha_2(x - ae_1) < \pi.$$

(Again the choice of $\alpha_1(x-ae_2)$ is unique.) We continue this process in both the upward and downward directions and then move on to the next lattice site on the horizontal axis. The angles of the vertical links are not affected by this construction and always lie in $[-\pi, \pi]$.

3.10.2 Deformations of Lattice Gauge Fields

We want to show that the trace of $H_w/\sqrt{H_w^2}$ is unchanged when we halve the lattice. To do that, we'd like to continuously deform the divided lattice gauge field to something simpler, which contains a lot of zeros and resembles the original lattice gauge field. The intuitive idea is illustrated in Figure 3.5. In particular, a plaquette will be deformed

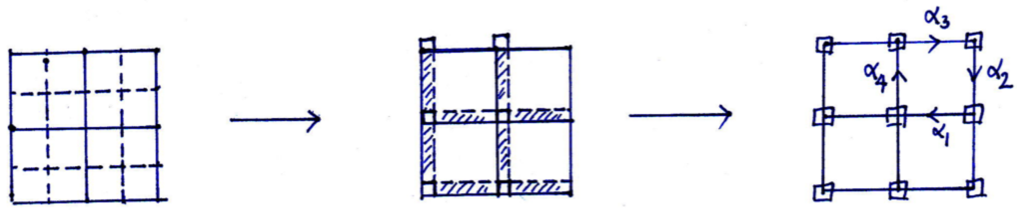

Figure 3.5: Deformation of the lattice

as in Figure 3.6. There are two conditions that need to be satisfied throughout the deformation: the topological charge Q is unchanged and $\max|1 - U(x)| < \varepsilon$. The condition $\max|1-U(x)| < \varepsilon$ guarantees that the operator H_w will have no zero eigenvalue throughout the deformation and so $H_w/\sqrt{H_w^2}$ is well-defined. Furthermore, the trace of $H_w/\sqrt{H_w^2}$ will change continuously and since it is an integer, it will be unchanged throughout the deformation. The simplest deformation is the straght line deformation, specifically:

$$\alpha_\mu(x, t) = (1 - t)\alpha_\mu(x, 0) + t\alpha_\mu(x, 1), \quad 0 \le t \le 1.$$

To check the two conditions, it suffices to look at one plaquette at a particular time t between 0 and 1, as illustrated in Figure 3.7.

First we compute the sum of the angles around each subplaquette. For the first sub-

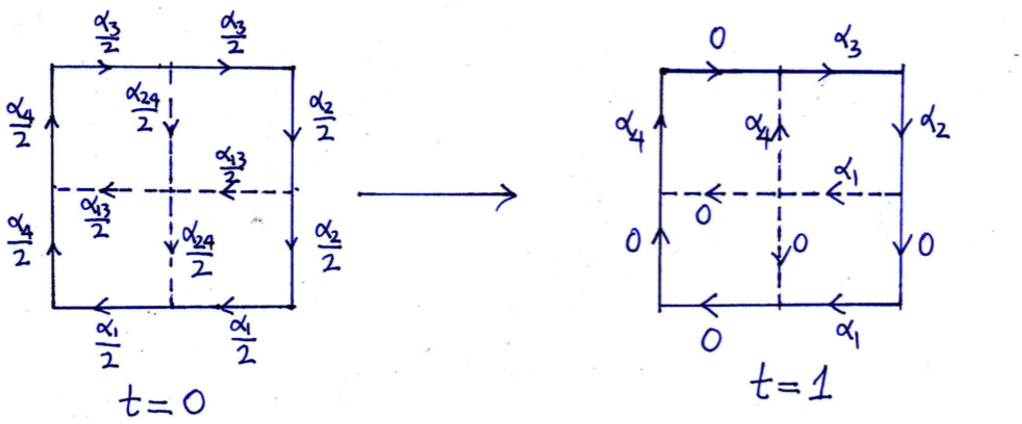

Figure 3.6: Deformation of a plaquette

plaquette:

$$\begin{aligned}
&\left[(1-t)\frac{\alpha_1}{2}+t\alpha_1\right]+(1-t)\frac{\alpha_2}{2}-\left[(1-t)\frac{\alpha_{13}}{2}+t\alpha_1\right]-(1-t)\frac{\alpha_{24}}{2}\\
&=\left(\frac{t+1}{2}\right)\alpha_1+(1-t)\frac{\alpha_2}{2}-(1-t)\left(\frac{\alpha_1-\alpha_3}{4}\right)-t\alpha_1-(1-t)\left(\frac{\alpha_2-\alpha_4}{4}\right)\\
&=\left(\frac{1-t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4).
\end{aligned}$$

For the second subplaquette:

$$\begin{aligned}
&\left[(1-t)\frac{\alpha_{13}}{2}+t\alpha_1\right]+\left[(1-t)\frac{\alpha_2}{2}+t\alpha_2\right]+\left[(1-t)\frac{\alpha_3}{2}+t\alpha_3\right]-\left[(1-t)\frac{\alpha_{24}}{2}-t\alpha_4\right]\\
&=(1-t)\left(\frac{\alpha_1-\alpha_3}{4}\right)+t\alpha_1+\left(\frac{1+t}{2}\right)\alpha_2+\left(\frac{1+t}{2}\right)\alpha_3-(1-t)\left(\frac{\alpha_2-\alpha_4}{4}\right)+t\alpha_4\\
&=\left(\frac{1+3t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4).
\end{aligned}$$

For the third subplaquette:

$$\begin{aligned}
&(1-t)\frac{\alpha_{13}}{2}+\left[(1-t)\frac{\alpha_{24}}{2}-t\alpha_4\right]+(1-t)\frac{\alpha_3}{2}+\left[(1-t)\frac{\alpha_4}{2}-t\alpha_4\right]\\
&=(1-t)\left(\frac{\alpha_1-\alpha_3}{4}\right)+(1-t)\left(\frac{\alpha_2-\alpha_4}{4}\right)-t\alpha_4+(1-t)\frac{\alpha_3}{2}+(1-t)\frac{\alpha_4}{2}+t\alpha_4\\
&=\left(\frac{1-t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4).
\end{aligned}$$

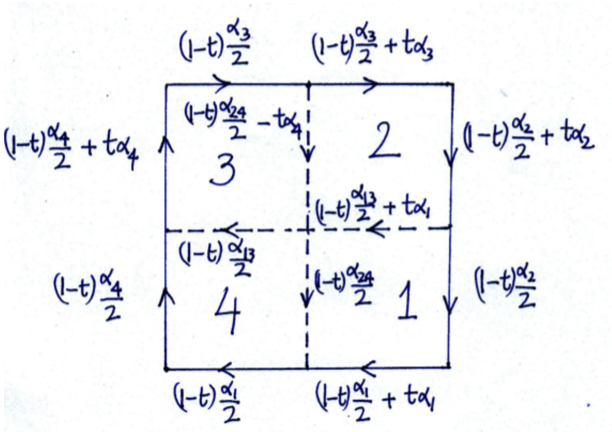

Figure 3.7: A plaquette at a particular time t

For the fourth subplaquette:

$$
\begin{aligned}
&(1-t)\frac{\alpha_1}{2}+(1-t)\frac{\alpha_{24}}{2}-(1-t)\frac{\alpha_{13}}{2}+(1-t)\frac{\alpha_4}{2}\\
&=(1-t)\frac{\alpha_1}{2}+(1-t)\left(\frac{\alpha_2-\alpha_4}{4}\right)-(1-t)\left(\frac{\alpha_1-\alpha_3}{4}\right)+(1-t)\frac{\alpha_4}{2}\\
&=\left(\frac{1-t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4).
\end{aligned}
$$

Since $0\le t\le 1$, we have

$$0\le\frac{1-t}{4}\le\frac{1}{4}\quad\text{and}\quad\frac{1}{4}\le\frac{1+3t}{4}\le 1.$$

Thus

$$\left|\left(\frac{1-t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right|<\frac{\pi}{4},$$

and

$$\left|\left(\frac{1+3t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right|<\pi.$$

Hence $\log U(x)$ is precisely the sum of the angles around each subplaquette. If we sum the angles of all the subplaquettes, we obtain $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, and so Q stays unchanged. Finally, since $-\pi<\alpha_1+\alpha_2+\alpha_3+\alpha_4<\pi$,

$$\left|1-\exp\left[i\left(\frac{1-t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right]\right|<\left|1-e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\right|<\varepsilon,$$

and

$$\left|1-\exp\left[i\left(\frac{1+3t}{4}\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\right]\right|<\left|1-e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}\right|<\varepsilon.$$

That shows $\max|1-U(x)|<\varepsilon$.

3.10.3 Index Density

Consider the vector space W of functions from the lattice sites to $\mathbb{C}$ with periodic boundary condition and A is an operator on it. We know that an orthonormal basis for W is given by

$$\beta = \left\{ \frac{1}{a}\delta_x \colon x \in \Gamma \right\}.$$

If ψ is an element in W, then it can be expressed as

$$\psi = \sum_{x\in\Gamma} a\psi(x)\frac{\delta_x}{a}.$$

Thus with respect to β, ψ is represented by a vector whose entries are the values of ψ on the lattice sites in Γ. Let A be the matrix representation of A with respect to β. The entries of A can be labelled by the lattice sites and we let $A(x,y)$ denote $1/a^2$ the (x,y)-entry of A, i.e.,

$$A(x,y) = \frac{1}{a^2}\left\langle A\frac{\delta_y}{a}, \frac{\delta_x}{a} \right\rangle.$$

With that notation, we have

$$(A\psi)(x) = a^2 \sum_{y\in\Gamma} A(x,y)\psi(y), \quad x \in \Gamma.$$

In our case, ψ is a function from the lattice sites to $\mathbb{C}^2$. The basis element δ_x acquires another index i

$$\delta_x^i = \delta_x e_i.$$

By representing A with respect to the new basis, taken into account the new index, we obtain the same formula

$$(A\psi)(x) = a^2 \sum_{y\in\Gamma} A(x,y)\psi(y), \quad x \in \Gamma.$$

However in this case $A(x,y)$ is a 2×2 matrix, i.e., an operator on $\mathbb{C}^2$. The (i,j)-entry of that matrix is given by

$$A(x,y)_{ij} = \frac{1}{a^2}\left\langle A\frac{\delta_y^j}{a}, \frac{\delta_x^i}{a} \right\rangle.$$

The trace of A is obtained by taking the sum of the diagonal entries of A, thus

$$\operatorname{tr} A = a^2 \sum_{x\in\Gamma} \operatorname{tr} A(x,x).$$

Recall that the index of D is given by

$$\text{Index}\,(D)_{\text{lattice}} = -\frac{1}{2}\operatorname{tr}\frac{H_w(1/a)}{\sqrt{H_w^2(1/a)}} = -\frac{1}{2}a^2\sum_{x\in\Gamma}\operatorname{tr}\frac{H_w(1/a)}{\sqrt{H_w^2(1/a)}}(x,x).$$

The term

$$q(x) = -\frac{1}{2}\mathrm{tr}\,\frac{H_w(1/a)}{\sqrt{H_w^2(1/a)}}(x,x)$$

is called an *index density*. The formula for the index becomes

$$\mathrm{Index}\,(D) = a^2 \sum_{x\in\Gamma} q(x).$$

The reason we want to express the index in this form is because in the continuum, the index (which is the same as the topological charge) is

$$\frac{1}{2\pi}\int_{[0,1]^2} F_{12}(x_1,x_2)dx_1dx_2.$$

If we define the *continuum index density* as

$$q^A(x) = \frac{1}{2\pi}F_{12}(x),$$

then the continuum index becomes

$$\lim_{a\to 0} a^2 \sum_{x\in\Gamma} q^A(x),$$

which resembles our formula in the discrete case.

3.10.4 Computational Approaches

In this subsection we present some computational approaches to the index of D. Since the index involves $1/\sqrt{H_w^2}$, the key is to find different ways to express the inverse square root of H_w^2 into a more manageable form. (In a computer, we can compute the eigenvalues of H_w for certain gauge fields approximately.) We're going to discuss two ways to represent the inverse square root, one via integral representation and one via Legendre polynomials (Section 1.4).

We first talk about integration of matrices. Consider a family of operators $\{A(x)\}$ on V depending smoothly on x, i.e., if we choose a basis for V, then $\{A(x)\}$ becomes a family of matrices and each entry is a smooth function in x. This definition of smoothness doesn't depend on the choice of basis since if we change the basis, then each matrix $\mathsf{A}(x)$ becomes $Q^{-1}\mathsf{A}(x)Q$, where Q is some invertible matrix not depending on x. We define

$$\int A(x)dx \colon = \int \mathsf{A}(x)dx,$$

where the integral on the right hand side is applied to each entry of the matrix $\mathsf{A}(x)$. Again this definition doesn't depend on the choice of basis since it can be shown easily that

$$\int Q^{-1}\mathsf{A}(x)Qdx = Q^{-1}\left(\int \mathsf{A}(x)dx\right)Q$$

for some invertible matrix Q not depending on x.

Proposition 3.15. *Let A be a strictly positive operator (the eigenvalues are positive), the inverse square root $A^{-1/2}$ has the following integral representation*

$$A^{-1/2} = \frac{2}{\pi}\int_0^\infty (A+t^2)^{-1}dt.$$

Proof. Let $(v_1, ..., v_n)$ is an orthonormal basis for V consisting of eigenvalues of A, i.e.,

$$Av_i = \lambda_i v_i, \quad i = 1, ..., n.$$

The eigenvalues of $(A + t^2)^{-1}$ are smooth functions of t:

$$(\lambda_i + t^2)^{-1}, \quad i = 1, ..., n.$$

Thus to show the equality, it suffices to show that

$$\frac{1}{\sqrt{\lambda}} = \frac{2}{\pi}\int_0^\infty \frac{dt}{\lambda + t^2}.$$

We evaluate the integral of the right hand side:

$$\begin{aligned}\int_0^\infty \frac{dt}{t^2+\lambda} &= \int_0^\infty \frac{dt}{\lambda\left(\left(\frac{t}{\sqrt{\lambda}}\right)^2+1\right)} \\ &= \frac{1}{\sqrt{\lambda}}\int_0^\infty \frac{d\left(\frac{t}{\sqrt{\lambda}}\right)}{\left(\frac{t}{\sqrt{\lambda}}\right)^2+1} \\ &= \frac{1}{\sqrt{\lambda}}\arctan\left(\frac{t}{\sqrt{\lambda}}\right)\Bigg|_0^\infty \\ &= \frac{\pi}{2\sqrt{\lambda}}.\end{aligned}$$

□

Now we present another way to express $1/\sqrt{H_w^2}$, via the Legendre polynomials. Recall that the Legendre polynomials have the following generating function:

$$\frac{1}{\sqrt{x^2 - 2tx + 1}} = \sum_{n=0}^\infty P_n(t)x^n, \quad |x| < 1,\ |t| \le 1.$$

From Section 3.6 we know that there exists a constant u independent of the choice of the gauge field such that

$$H_w^2 \ge u > 0$$

provided the gauge fields satisfy the approximate smoothness condition

$$\max_{x,\mu,\nu} |1 - U(x,\mu,\nu)| < \varepsilon$$

for some appropriately chosen $\varepsilon > 0$. (Here the inequality of H_w means the corresponding inequalities for its eigenvalues.) The operator H_w also has an upper bound independent of the gauge fields as follows:

$$\|H_w\| = \left\|\gamma_3\left(D_w - \frac{1}{a}\right)\right\| \le \sum \|\nabla_\mu\| + \frac{a}{2}\sum \|\Delta_\nu\| + \frac{1}{a} = \frac{9}{a}.$$

(Consider Section 3.6 for the bounds used for ∇ and Δ.) Let $v > u$ be an upper bound for H_w^2 and consider the following operator

$$T = \frac{1}{v-u}(v + u - 2H_w^2).$$

The operator T is self-adjoint. If λ is an eigenvalue of H_w^2, then

$$u \le \lambda \le v.$$

Therefore,

$$u - v \le v + u - 2\lambda \le v - u.$$

Thus the eigenvalues of T lies in $[-1, 1]$. It follows that

$$\frac{1}{\sqrt{x^2 - 2xT + 1}} = \sum_{n=0}^{\infty} x^n P_n(T), \quad |x| < 1. \tag{3.7}$$

(This is true because T is hermitian and the equality holds for all eigenvalues of T.) The expansion is true for any $|x| < 1$ and so we can choose x to simplify it. Since

$$\frac{v+u}{v-u} > 1,$$

there exists a unique $\theta > 0$ such that

$$\frac{v+u}{v-u} = \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}.$$

Put $x = e^{-\theta}$, then clearly $|x| < 1$ and we have

$$x^2 - 2x\left(\frac{v+u}{v-u}\right) + 1 = e^{-2\theta} - 2e^{-\theta}\left(\frac{e^{\theta} + e^{-\theta}}{2}\right) + 1 = 0.$$

With that choice of x, the expression on the left hand side of (3.7) becomes

$$\begin{aligned} x^2 - 2xT + 1 &= x^2 - 2x\frac{1}{v-u}(v + u - 2H_w^2) + 1 \\ &= x^2 - 2x\left(\frac{v+u}{v-u}\right) + 1 + \frac{4x}{v-u}H_w^2 \\ &= \frac{4x}{v-u}H_w^2. \end{aligned}$$

We finally obtain an expansion of $1/\sqrt{H_w^2}$ in terms of the Legendre polynomials:

$$\frac{1}{\sqrt{H_w^2}} = \sqrt{\frac{4x}{v-u}}\sum_{n=0}^{\infty} x^n P_n(T).$$

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