THREE-MANIFOLD CONSTRUCTIONS AND CONTACT STRUCTURES

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1. Introduction

Recently, there's been an interest in open book decompositions of 3-manifolds, partly because of a fundamental theorem by Giroux, which provides a correspondence between contact structures and open book decompositions of a 3-manifold. The theorem leads to various topological applications of contact geometry. In this essay, we give a brief introduction to the open book decomposition of a 3-manifold, starting from Heegaard diagrams and ending with Thurston's proof of the fact that every open book decomposition of a 3-manifold supports a contact structure. We only assume very basic topology and mostly appeal to the intuition of the readers. Many figures are given to aid in this direction. Our focus is in examples, which serve to illustrate many useful topological descriptions of 3-manifolds: Heegaard diagrams, surgery, branched covering, open book decomposition. The exposition mostly follows the beautiful books [12] and [11]. It is hoped that the essay will whet the readers' appetite and motivate them to explore the wonderful world of low-dimensional topology.

2. Heegaard Diagrams

Given two manifolds M and N with a homeomorphism $h\colon \partial N \to \partial M$, we can obtain a new manifold without boundary, denoted $M \cup_h N$, by gluing M and N along their boundaries via the homeomorphism h. Formally, $M \cup_h N$ is given by

$$M \cup_h N = M \sqcup N / \sim,$$

where $\sim$ is the equivalence relation $x \sim h(x)$ for $x \in \partial N$. Recall that two homeomorphisms $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are *isotopic* if there is a homeomorphism

$$F\colon X \times [0,1] \to Y \times [0,1]$$

such that $F(x,t) = (f(x,t),t)$, $x \in X$, $t \in [0,1]$, with $f(x,0) = f_0(x)$ and $f(x,1) = f_1(x)$. We denote $f(x,t)$ by $f_t(x)$. Our key result regarding gluing is the following:

Proposition 1. *Suppose that M and N are two manifolds with homeomorphic boundaries, and that $h_0\colon \partial N \to \partial M$ and $h_1\colon \partial N \to \partial M$ are isotopic homeomorphisms. Then $M \cup_{h_0} N$ and $M \cup_{h_1} N$ are homeomorphic.*

Proof. Let $h_t\colon \partial N \to \partial M$ be a family of homeomorphisms connecting h_0 and h_1 and C a collar neighborhood of ∂N in N, i.e. C is a neighborhood of ∂N homeomorphic to $\partial N \times [0,1]$, with ∂N identified with $\partial N \times \{0\}$. We can construct a map $f\colon M \cup_{h_0} N \to M \cup_{h_1} N$ by defining f to be the identity on $M \cup (N - C)$ and on C defining $f(x,t) = (h_1^{-1}h_t(x), t)$. We need to check that f is well-defined, that is $f(x,1) = (x,1)$ and $f(x) = f(h_0(x))$ for $x \in \partial N$. To see the first condition, note that

$$f(x,1) = (h_1^{-1}h_1(x), 1) = (x,1).$$

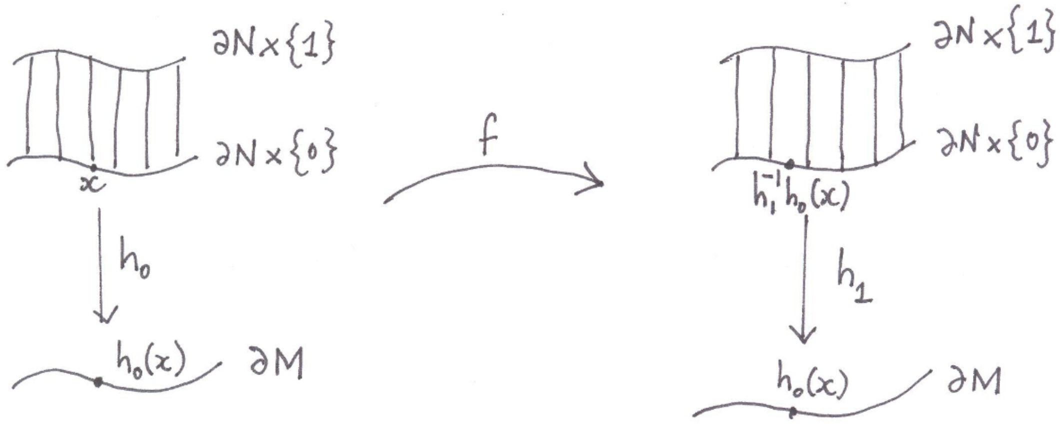
Advisor: Professor Dale Rolfsen.

For the second condition, we have

$$f(x) = h_1^{-1}h_0(x) = h_0(x) = f(h_0(x)),$$

where the second equality follows from the identification $h_1\colon \partial N \to \partial M$ and the third equality follows because f is the identity on M (see Figure 1). Thus f is a homeomorphism. $\square$

FIGURE 1. The map f is well-defined



Lemma 1 (Alexander Trick). *Every homeomorphism $h\colon \mathbb{S}^n \to \mathbb{S}^n$ can be extended to a homeomorphism $H\colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$, where $\mathbb{D}^{n+1}$ is a unit ball in $\mathbb{R}^{n+1}$.*

Proof. We can define a homeomorphism H as follows:

$$H(tx) = tH(x),$$

where $x \in \mathbb{S}^n$ and $0 \le t \le 1$. $\square$

Theorem 1. *Suppose $M = D_1 \cup_h D_2$ is obtained by gluing two $(n+1)$-disks $D_1 = D_2 = \mathbb{D}^{n+1}$ along their boundaries via a homeomorphism $h\colon \mathbb{S}^n \to \mathbb{S}^n$, then M is homeomorphic to $\mathbb{S}^{n+1}$.*

Proof. Recall that $\mathbb{S}^{n+1}$ is obtained by gluing two $(n+1)$-disks along their boundaries via the identity map. Extend h to a homeomorphism H as in the preceding lemma. We can therefore construct a map f from M onto $\mathbb{S}^{n+1}$ as follows

$$f(x) = \begin{cases} x & \text{for } x \in D_1, \\ H(x) & \text{for } x \in D_2. \end{cases}$$

Clearly f is consistent on the boundary $\mathbb{S}^n$ since if $x = h(y)$ for $x \in D_1$ and $y \in D_2$, then

$$f(x) = x = h(y) = f(y).$$

Thus f is a homeomorphism. $\square$

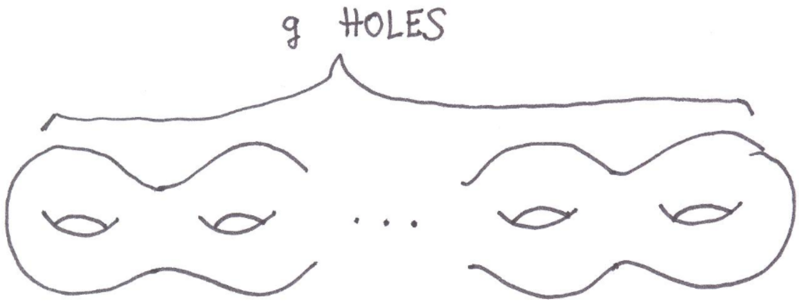
By essentially the same idea we have the following proposition.

Proposition 2. *Suppose $\mathbb{D}^{n+1}$ is an embedded submanifold of M. Then $(M - \mathbb{D}^{n+1}) \cup_h \mathbb{D}^{n+1}$ is homeomorphic to M, where h is any homeomorphism $\mathbb{S}^n \to \mathbb{S}^n$.*

Proof. Proceed as in the proof of Theorem 1, but replace D_1 by $M - \mathbb{D}^{n+1}$. $\square$

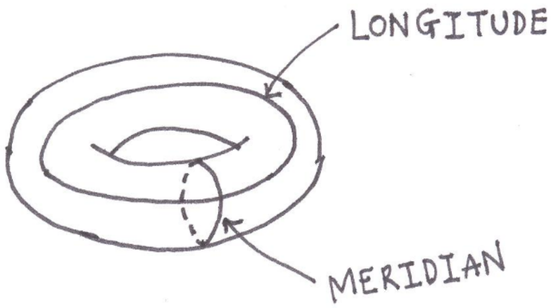
Definition 1. A *handlebody* of genus g is the result of attaching g disjoint 1-handles $\mathbb{D}^2 \times [-1,1]$ to a 3-ball by gluing the parts $\mathbb{D}^2 \times \{\pm 1\}$ to $2g$ disjoint disks on the boundary $\mathbb{S}^2$ in such a way that the result is an orientable 3-manifold with boundary (see Figure 2). Note that the boundary of a handlebody of genus g is a closed orientable surface of genus g.

FIGURE 2. The standard embedding of a handlebody in $\mathbb{R}^3$



We call a handlebody T of genus 1 a *solid torus* (see Figure 3). Note that a solid torus is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. A *meridian* of T is a curve on ∂T that bounds a disk in T and does not bound a disk in ∂V. A *longitude* of T is any simple closed curve on ∂V of the form $h(\mathbb{S}^1 \times 1)$, where $h\colon \mathbb{S}^1 \times \mathbb{D}^2 \to T$ is a homeomorphism.

FIGURE 3. The standard embedding of a solid torus in $\mathbb{R}^3$



Lemma 2. *Every finite connected embedded graph in an oriented 3-manifold gives us a handlebody.*

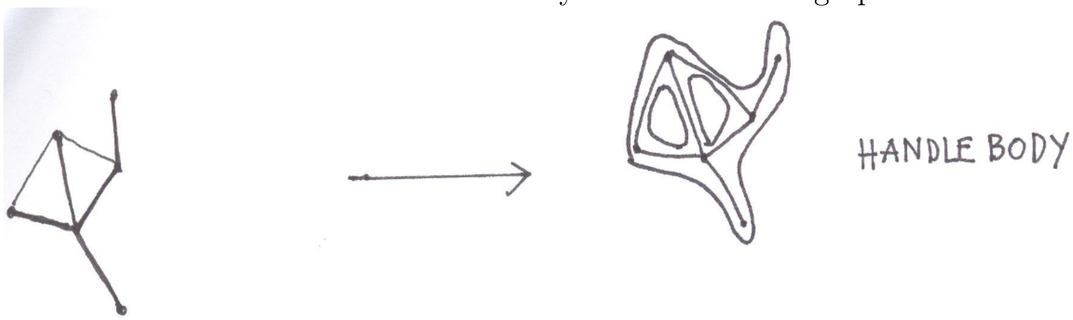
Proof. If M is an oriented 3-manifold, every finite connected graph Γ embedded in M admits a tubular neighborhood ν_Γ (i.e. ν_Γ is an open neighborhood of Γ in M such that $\pi\colon \nu_\Gamma \to \Gamma$ is an oriented vector bundle). It turns out that ν_Γ is a handlebody. This follows by taking the neighborhood of a maximal tree as the 3-ball and neighborhoods of the remaining edges as 1-handles (see Figure 4). $\square$

Definition 2. A *Heegaard splitting* of genus g of a 3-manifold M is a pair of genus g handlebodies X and Y contained in M such that $M = X \cup Y$ and $X \cap Y = \partial X = \partial Y$.

Proposition 3. *Any closed, connected, orientable 3-manifold M has a Heegaard splitting.*

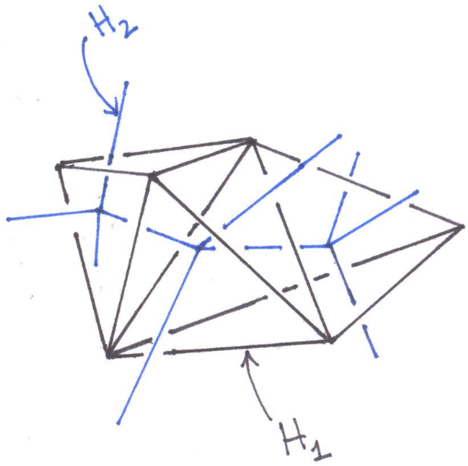
Proof. First we triangulate the 3-manifold M. Since M is closed, the triangulation is finite. From Lemma 2, we just need to find two finite connected graphs embedded in M. The one-skeleton of M is a finite connected graph and that gives us a handlebody H_1.

Figure 4. The handlebody obtained from a graph



To obtain the second handlebody, we place a vertex at the center of each tetrahedron. If two tetrahedra are glued together along a face, put in an edge connecting the two vertices at the centers of the tetrahedra, passing through the shared face. That gives us another finite connected graph embedded in M. Now if we thicken up this graph enough, we'll obtain another handlebody H_2 that fills up the remaining part of M and shares a common boundary with H_1 (see Figure 5). That gives us a Heegaard splitting of M. $\square$

Figure 5. The graph H_1 and its dual H_2



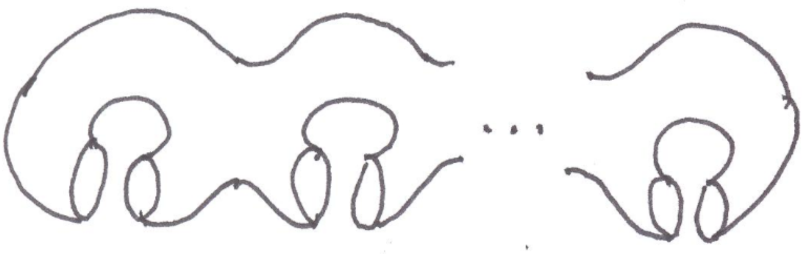
Heegaard splitting gives us a useful way to represent closed, connected, orientable 3-manifolds in terms of handlebodies. Suppose $M = H_1 \cup_h H_2$ is a genus g Heegaard splitting of M. We assume that H_2 is standardly embedded in $\mathbb{S}^3$. Let $D_1, ..., D_g$ be the disks of H_2 corresponding to the centers $D_i \times \{0\}$ of the 1-handles $D_i^2 \times [-1, 1]$ (see Figure 6). Let C_i be the boundary of D_i and let $l_i = h(C_i)$ the image of C_i on ∂H_1. The handlebody H_1 together with the collection of curves l_i is called a *Heegaard diagram* of genus g for M.

Thus a closed, connected, orientable 3-manifold M gives rise to a Heegaard diagram $(H_1, \{l_1, ..., l_g\})$ where H_1 is a handlebody of genus g and $\{l_1, ..., l_g\}$ are disjoint closed curves on ∂H_1. Let N_i be a collar neighborhood of l_i in ∂H_1, i.e. $N_i = l_i \times [-1/2, 1/2]$. We call (the closure of) $\partial H_1 - \cup_i N_i$ the result of *cutting* ∂H_1 *along* $\{l_1, ..., l_g\}$. Notice that $\partial H_1 - \cup_i N_i$ is homeomorphic to the result of cutting ∂H_2 along $\{C_1, ..., C_g\}$, which is a 2-sphere with $2g$ disks removed (see Figure 7).

Conversely, given $(H_1, \{l_1, ..., l_g\})$ where H_1 is a handlebody of genus g and $\{l_1, ..., l_g\}$ are disjoint closed curves on ∂H_1 such that $\partial H_1 - \cup_i N_i$ is a 2-sphere with $2g$ disks removed, we can show that it is a Heegaard diagram of a 3-manifold M as follows. Consider a

FIGURE 6. The collection of disks of H_2

FIGURE 7. A sphere with $2g$ disks removed



handlebody H_2 standardly embedded in $\mathbb{S}^3$. First we glue $D_i \times [-1/2, 1/2]$ to N_i by a homeomorphism that maps C_i to l_i and extends by the identity in the t coordinate, $t \in [-1/2, 1/2]$. Since $\partial H_1 - \bigcup_i N_i$ is homeomorphic to a 2-sphere with $2g$ disks removed, choose a homeomorphism between $\partial H_1 - \bigcup_i N_i$ and $\partial H_2 - \bigcup_i D_i \times [-1/2, 1/2]$. We thus get a manifold M obtained from gluing two handlebodies H_1 and H_2 via a homeomorphism that sends C_i to l_i.

One might wonder whether the construction above can give us different manifolds from the same Heegaard diagram. The answer is provided in the next proposition.

Proposition 4. *Suppose $(H_1, \{l_1, ..., l_g\})$ is a Heegaard diagram of a 3-manifold M and $(H_1', \{l_1', ..., l_g'\})$ is a Heegaard diagram of a 3-manifold M'. If there is a homeomorphism $\phi\colon H_1 \to H_1'$ taking each l_i onto l_i'. Then M and M' are homeomorphic.*

Proof. Suppose $M = H_1 \cup_h H_2$ and $M' = H_1' \cup_{h'} H_2'$. We want to construct a homeomorphism $M \to M'$. The homeomorphism $\phi\colon H_1 \to H_1'$ gives us a homeomorphism $\phi\colon \partial H_2 \to \partial H_2'$ that sends l_i on ∂H_2 onto l_i' on $\partial H_2'$. Note that each l_i bounds a disk D_i in H_2 and each l_i' bounds a disk D_i' in H_2'. To extend ϕ to a homeomorphism $H_2 \to H_2'$, we proceed as follows.

First we extend $\phi\colon l_i \to l_i'$ to a homeomorphism $D_i \to D_i'$ using Lemma 1. We then extend ϕ to a homeomorphism $D_i \times [-1/2, 1/2] \to D_i' \times [-1/2, 1/2]$ via the identity on the second factor. We are now left with a homeomorphism

$$\phi\colon \partial(H_2 - \bigcup_i D_i \times (-1/2, 1/2)) \to \partial(H_2' - \bigcup_i D_i' \times (-1/2, 1/2)).$$

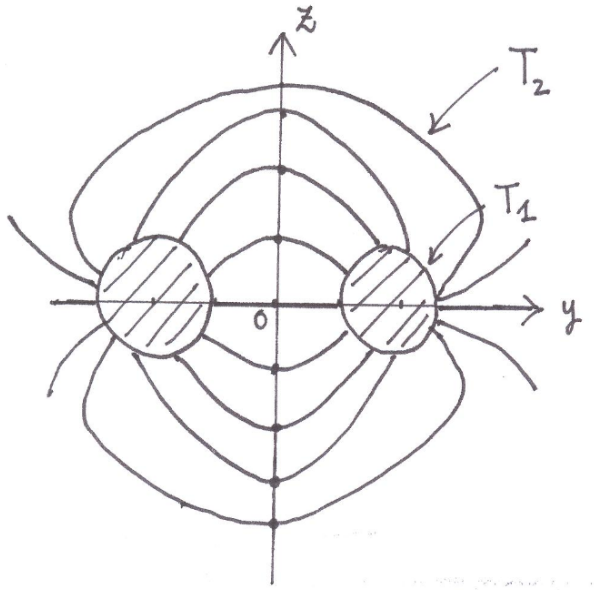
Notice that the domain and target manifolds are the 2-spheres, by the definition of Heegaard diagram. Now we just need to extend ϕ to a homeomorphism of the 3-balls whose boundaries are the given 2-spheres by another application of Lemma 1. $\square$

Remark 1. This proposition says that starting from a Heegaard diagram, we obtain a unique manifold up to homeomorphism. More generally, if $(H_1, \{l_1, ..., l_g\})$ and $(H_1, \{l'_1, ..., l'_g\})$ are isotopic Heegaard diagrams on the same handlebody, i.e. there exists an isotopy of ∂H_1 taking each l_i onto l'_i, then the manifolds obtained from these Heegaard diagrams are homeomorphic. To see why, suppose $M = H_1 \cup_h H_2$ and $M' = H_1 \cup_{h'} H_2$. By composing h with the isotopy, we obtain a map $h'' \colon \partial H_1 \to \partial H_2$ isotopic to h that sends each l_i onto l'_i. By Proposition 1, $M'' = H_1 \cup_{h''} H_2$ is homeomorphic to $M = H_1 \cup_h H_2$. However, M'' and M' are homeomorphic because they have the same Heegaard diagrams by the above proposition. Therefore M and M' are homeomorphic. In particular, a manifold obtained from gluing two solid tori is completely determined by a simple closed curve (up to isotopy) on the boundary of the solid torus.

Example 1 (Heegaard diagrams of $\mathbb{S}^3$)**.** In this example, we show that $\mathbb{S}^3$ has a Heegaard splitting into handlebodies of arbitrary genus. We've seen that the sphere $\mathbb{S}^3$ can be expressed as a union of two 3-balls with a common boundary. We'll describe several ways to see that $\mathbb{S}^3$ has a Heegaard splitting into handlebodies of genus 1, i.e. solid tori.

(1) The sphere $\mathbb{S}^3$ is the compactification $\mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$. The first solid torus T_1 can be obtained by rotating the disk $(y-2)^2 + z^2 \leq 1$ around the z-axis. The complement of T_1 can be viewed as a family of disks parametrized by the circle $Oz \cup \{\infty\}$, thereby forming a solid torus T_2. These two solid tori share a common boundary torus (see Figure 8).

FIGURE 8. The sphere $\mathbb{S}^3$ as a union of two solid tori



(2) For a more precise description, we can consider the 3-sphere as the following subset of $\mathbb{C}^2$:

$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \colon |z|^2 + |w|^2 = 1\}.$$

We can decompose $\mathbb{S}^3$ into

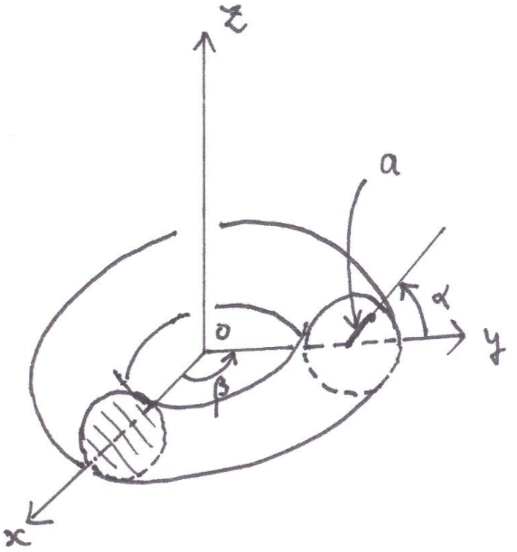
$$T_1 = \{(z, w) \in \mathbb{S}^3 \colon |z| \leq |w|\}, \quad T_2 = \{(z, w) \in \mathbb{S}^3 \colon |z| \geq |w|\}.$$

Note that in $\mathbb{S}^3$ the condition $|z| \leq |w|$ is equivalent to $|z| \leq 1/\sqrt{2}$ and the condition $|z| \geq |w|$ is equivalent to $|w| \leq 1/\sqrt{2}$. We'll show that T_1 and T_2 are solid tori.

A point in $\mathbb{S}^3$ can be presented as $(ae^{i\alpha}, be^{i\beta})$, where a, b are nonnegative real numbers, $a^2 + b^2 = 1$ and $\alpha, \beta \in [0, 2\pi]$. Therefore T_1 is determined by the condition $a \leq 1/\sqrt{2}$.

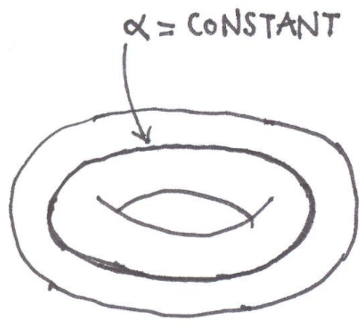
Consider the solid torus obtained by rotating the disk $(y-1)^2+z^2 \leq 1/2$ around the z-axis. We can put coordinates (a, α, β) on the solid torus as in Figure 9. The correspondence

FIGURE 9. Coordinates on the solid torus



$(ae^{i\alpha}, be^{i\beta}) \leftrightarrow (a, \alpha, \beta)$ then gives us a homeomorphism between T_1 and the solid torus. Similarly, we can obtain a homeomorphism of T_2 (determined by $b \leq 1/\sqrt{2}$) onto the solid torus by the correspondence $(ae^{i\alpha}, be^{i\beta}) \leftrightarrow (b, \beta, \alpha)$. Note that T_1 and T_2 intersect along the torus $(1/\sqrt{2}e^{i\alpha}, 1/\sqrt{2}e^{i\beta})$ (determined by $a = b = 1/\sqrt{2}$). The curve $\{\alpha = \text{constant}\}$ on the boundary torus is simultaneously a meridian of T_2 and a longitude of T_1. A Heegaard diagram of $\mathbb{S}^3$ is therefore given in Figure 10.

FIGURE 10. A Heegaard diagram of genus 1 for $\mathbb{S}^3$



(3) From another point of view, we show that if we glue two solid tori T_1 and T_2 along their boundary tori via a homeomorphism that sends each meridian of T_2 to a longitude of T_1, then we obtain $\mathbb{S}^3$. To see this, we glue T_2 to T_1 in two steps. First, we glue a neighborhood of a meridional disk of T_2 to T_1 via a homeomorphism which identifies a meridian of T_2 with a longitude of T_1. The resulting space B is a 3-ball (see Figure 11). We then glue the remaining piece of T_2, which is also a 3-ball, to B. But we know that the only space obtained by gluing two 3-balls along their boundaries is $\mathbb{S}^3$ (Theorem 1).

The procedure employed in (3) can be generalized to show that $\mathbb{S}^3$ can be obtained by gluing two handlebodies U and V of genus g via a homeomorphism f that identifies p_i and q_i, where p_i and q_i are given as in Figure 12. First we glue neighborhoods of disks

FIGURE 11. Fill in the hole of the solid torus T_1

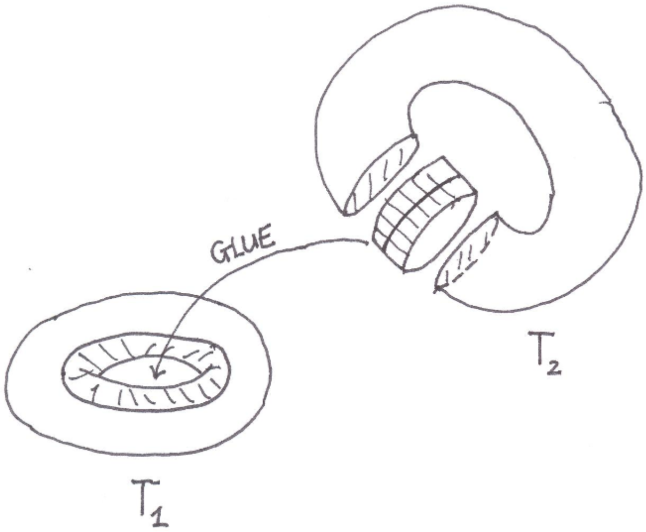
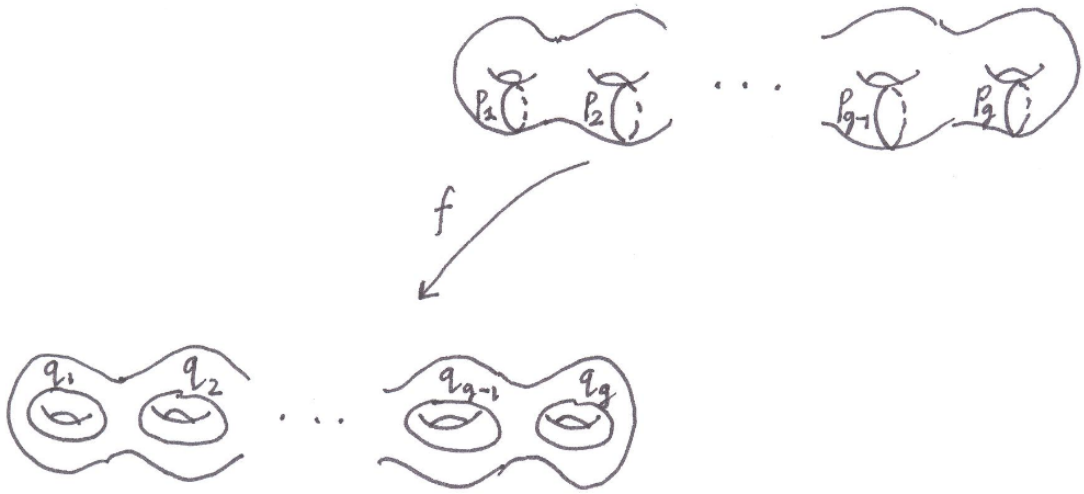


FIGURE 12. The curves p_i and q_j



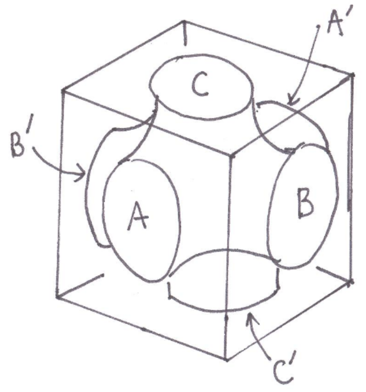
bounded by p_i's to U to obtain a 3-ball. The remaining part of V is also a 3-ball, and we know that the result of gluing two 3-balls along their boundaries is $\mathbb{S}^3$. Figure 13 gives a Heegaard diagram of genus g for $\mathbb{S}^3$.

FIGURE 13. A Heegaard diagram of genus g for $\mathbb{S}^3$

Example 2 (Heegaard diagram of the 3-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$)**.** The 3-torus T^3 can be obtained by gluing the opposite faces of the cube $[0,1] \times [0,1] \times [0,1]$ via the identity maps. To get a Heegaard splitting for T^3, we put a "six-legged" piece in the cube as

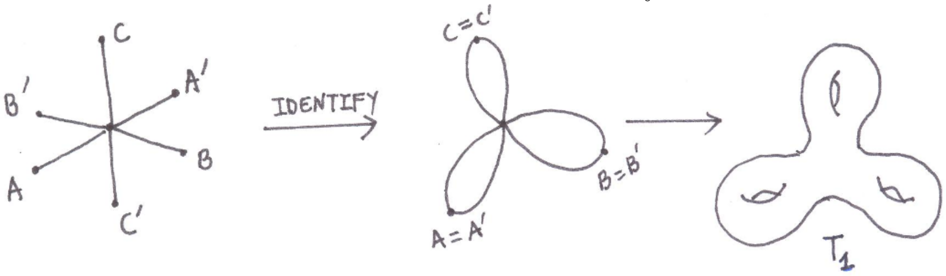
in Figure 14. The "six-legged" piece gives us a handlebody T_1 of genus 3 by identifying

Figure 14. A Heegaard splitting of the 3-torus



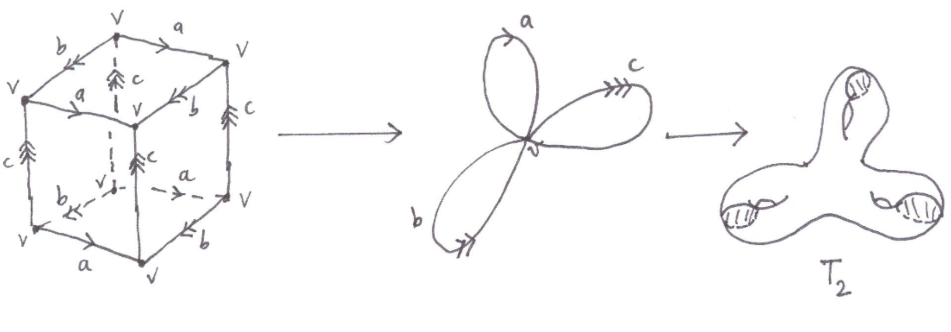
opposite faces (see Figure 15). The complement of the "six-legged" piece in the cube

Figure 15. The handlebody T_1



retracts to the edges of the cube, which gives us another handlebody T_2 of genus 3, see Figure 16.

Figure 16. The handlebody T_2



Now to find a Heegaard diagram for T^3, we need to identify the images of the boundaries of the meridional disks of the handlebody T_2. Note that a meridional disk of T_2 intersects T_1 as in Figure 17. To make it easier to draw a Heegaard diagram, we remove the 1-handles from T_1 and flatten the resulting $\mathbb{S}^2$ with 6 disks removed in the plane as in Figure 18. The identification between two disks of the same handle is indicated by the orientation (note that C and C' have the same orientation) . In the figure, it is understood that every time a line hits a disk, it goes along the 1-handle to the other disk.

FIGURE 17. The intersection of a meridional disk of T_2 with T_1

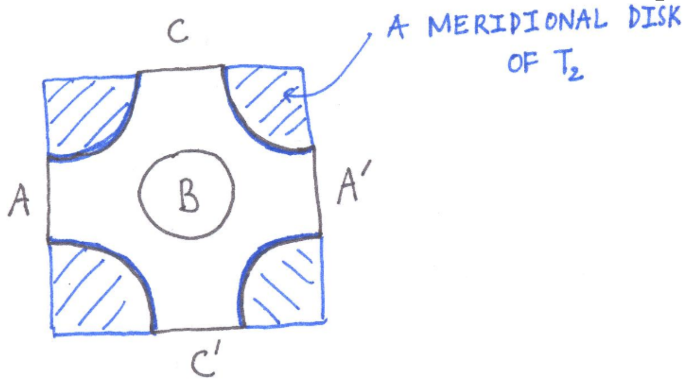
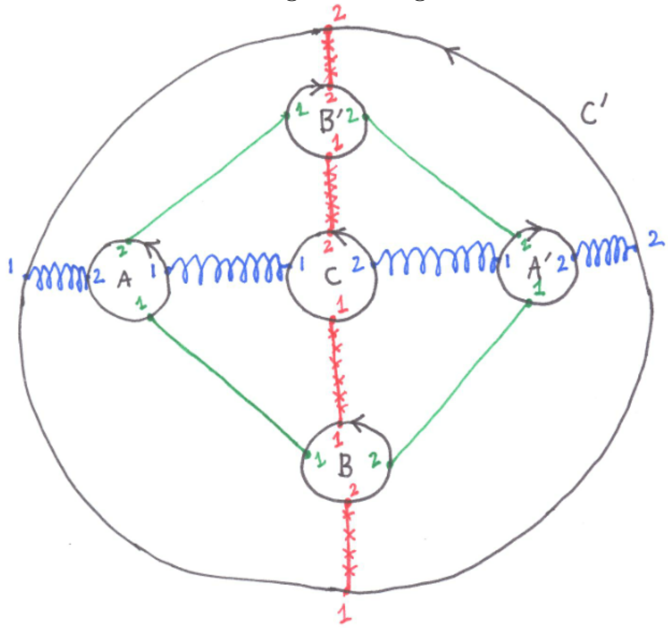
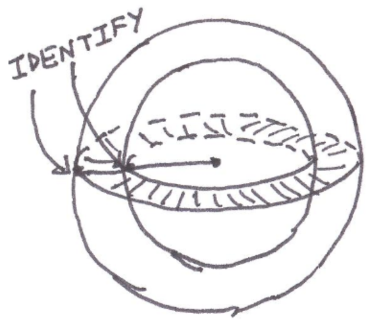


FIGURE 18. A Heegaard diagram of the 3-torus



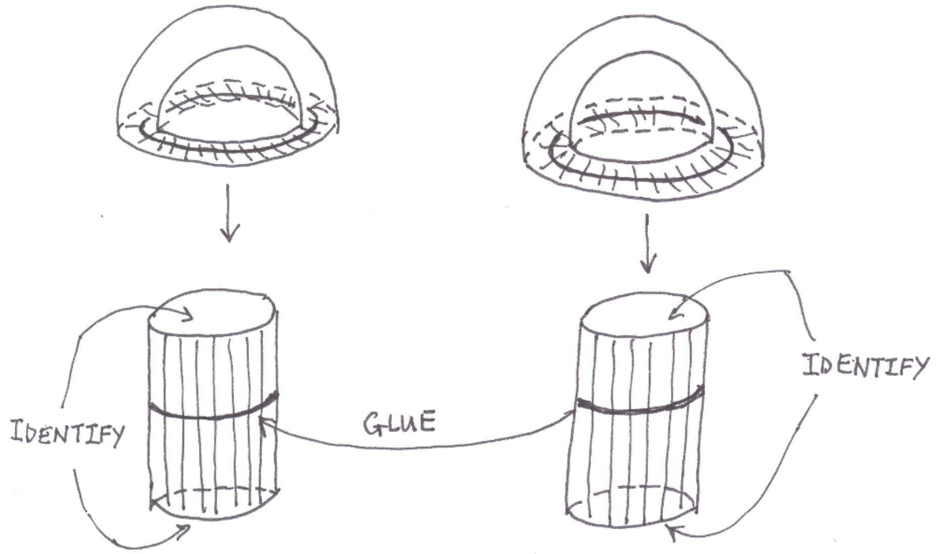
Example 3 (Heegaard diagram of $\mathbb{S}^1 \times \mathbb{S}^2$)**.** We can picture $\mathbb{S}^2 \times \mathbb{S}^1$ as follows. Starting with a 3-disk, remove a smaller 3-disk from its interior. Now glue the inner boundary sphere to the outer boundary sphere by identifying each point on the inside sphere to the point radially outward from it on the outside sphere (see Figure 19). To see a Heegaard

FIGURE 19. A way to visualize $\mathbb{S}^2 \times \mathbb{S}^1$



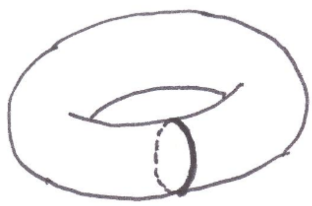
splitting of $\mathbb{S}^1 \times \mathbb{S}^2$, we cut $\mathbb{S}^1 \times \mathbb{S}^2$ into two pieces by a plane through the equators of the inner sphere and the outer spheres. Each piece is a solid torus (see Figure 20). Note that

FIGURE 20. A Heegaard splitting of $\mathbb{S}^2 \times \mathbb{S}^1$



the meridian of the first solid torus is glued to the meridian of the second solid torus. A Heegaard diagram of $\mathbb{S}^1 \times \mathbb{S}^2$ is therefore given in Figure 21.

FIGURE 21. A Heegaard diagram of $\mathbb{S}^2 \times \mathbb{S}^1$



Example 4 (Heegaard diagram of lens spaces)**.** Let p and q be coprime positive integers. Define an action of $\mathbb{Z}/p\mathbb{Z} = \langle \sigma \rangle$ on $\mathbb{S}^3 \subset \mathbb{C}^2$ as follows

$$\sigma(z, w) = (\exp(2\pi i/p)z, \exp(2\pi iq/p)w).$$

This action is fixed-point free, and so the quotient of $\mathbb{S}^3$ by this action is a 3-manifold. We call the resulting quotient space a lens space, denoted by $L(p, q)$.

Consider $\mathbb{S}^3$ as the following subset of $\mathbb{C}^2$

$$\mathbb{S}^3 = \{(|z| \exp(2\pi i\phi), |w| \exp(2\pi i\theta)) \colon |z|^2 + |w|^2 = 1\},$$

we obtain the following cell decomposition of $\mathbb{S}^3$ (by a cell we mean a space homeomorphic to the closed n-disk $\mathbb{D}^n$):

(0) 0-dimensional cells: $e^0_k = (0, \exp(2\pi ik/p))$,

(1) 1-dimensional cells: $e^1_k = (0, \exp(2\pi i\theta))$, $k/p \leq \theta \leq (k + 1)/p$,

(2) 2-dimensional cells: $e^2_k = (|z| \exp(2\pi ik/p), w)$, $0 \leq |z| \leq 1$, $|w| = \sqrt{1 - |z|^2}$,

(3) 3-dimensional cells: $e^3_k = (|z| \exp(2\pi i\phi), w)$, $0 \leq |z| \leq 1$, $k/p \leq \phi \leq (k + 1)/p$, $|w| = \sqrt{1 - |z|^2}$.

Here $k = 0, 1, ..., p-1$ and the gluing maps are inclusion. Notice that the action of $\mathbb{Z}/p\mathbb{Z}$ permutes the cells of each dimension with each other. Therefore the cell decomposition of the quotient space will consist of one cell in each dimension 0, 1, 2, 3. Thus to obtain $L(p,q)$, we can just pick any 3-cell e_k^3 and identify the points on its boundary according to the action of $\mathbb{Z}/p\mathbb{Z}$.

To better visualize the identification, we change the coordinates on e_k^3 to the natural coordinates (x_1,x_2,x_3) in $\mathbb{R}^3$. Under this change of coordinates, e_k^3 is just the usual 3-disk $\mathbb{D}^3 = \{x_1^2+x_2^2+x_3^2 \leq 1\}$. Specifically, to the point

$$(|z|\exp(2\pi i\phi), w),\ \text{where } 0 \leq |z| \leq 1,\ k/p \leq \phi \leq (k+1)/p,\ |w| = \sqrt{1-|z|^2}$$

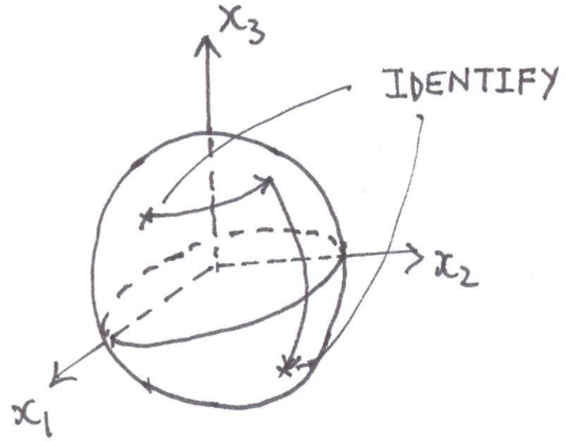
(note that this point is determined by $|z|$, ϕ and w) let us assign the point

$$(x_1,x_2,x_3) \in \mathbb{R}^3,\ \text{where } x_1+ix_2 = w,\ x_3 = (-2p\phi+2k+1)|z|.$$

Then $|x_3| = |-2p\phi+2k+1||z| \leq |z|$ because $k/p \leq \phi \leq (k+1)p$ and so $x_1^2+x_2^2+x_3^2 \leq |w|^2+|z|^2 = 1$. Therefore $(x_1,x_2,x_3) \in \mathbb{D}^3$. Conversely, given any point in $\mathbb{D}^3$, we can easily solve for w, ϕ, $|z|$ and thus recover the original point. A point (x_1,x_2,x_3) lies on the boundary $\mathbb{S}^2$ of $\mathbb{D}^3$ if and only if $|-2p\phi+2k+1| = 1$ if and only if $\phi = k/p$ or $\phi = (k+1)/p$. If $\phi = k/p$, then $x_3 = |z|$ and so (x_1,x_2,x_3) lies on the upper hemisphere of $\mathbb{S}^2$. If $\phi = (k+1)/p$, then $x_3 = -|z|$ and so (x_1,x_2,x_3) lies on the lower hemisphere of $\mathbb{S}^2$. (Note that the equator of $\mathbb{S}^2$ corresponds to $|z| = 0$.)

Now we can see the effect of the action of σ on $\mathbb{S}^2$. For a point on the upper hemisphere, σ will first rotate that point by an angle of $2\pi iq/p$ (effect of the term $\exp(2\pi iq/p)w$) and then identify the resulting point with its reflection about the equator of $\mathbb{S}^2$ (the term $\exp(2\pi i/p)z$ sends k/p to $(k+1)/p$). To summarize, we've obtained another description of $L(p,q)$ as a 3-disk $\mathbb{D}^3 \subset \mathbb{R}^3$ with certain points on $\mathbb{S}^2$ identified (see Figure 22). This description allows us to obtain a Heegaard splitting of $L(p,q)$ as a union of two solid tori.

FIGURE 22. The lens space as a quotient of $\mathbb{D}^3$



To see the Heegaard diagram of $L(p,q)$, we first decompose $\mathbb{D}^3$ as follows. Let V_1 be the intersection of $\mathbb{D}^3$ and the cylinder $x_1^2+x_2^2 \leq 1/2$ and V_2 is the closure of $\mathbb{D}^3 - V_1$. Under the identification on the boundary $\mathbb{S}^2$ defined as above, we need to glue the top and bottom of V_1 with a $2\pi q/p$ twist, thus obtain a solid torus T_1 (see Figure 23). To see V_2 under the identification is a solid torus, we cut it along p planes L_k, where L_k is a plane containing the x_3-axis with angle $2\pi k/p$ for $k = 1,...,p$, to obtain p polyhedra, as shown in Figure 24 (drawn for the case $p = 3$, $q = 2$). For each polyhedron, let T_i denote the top face, B_i denote the bottom face, L_i denote the left face and R_i denote the right face. Then we have the identification $R_i \leftrightarrow L_{i+1}$ and $T_i \leftrightarrow B_{i+q}$ for $i = 1,...,p$ (note that here addition is done modulo p).

FIGURE 23. The region V_1

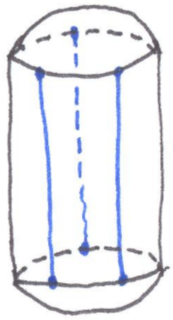
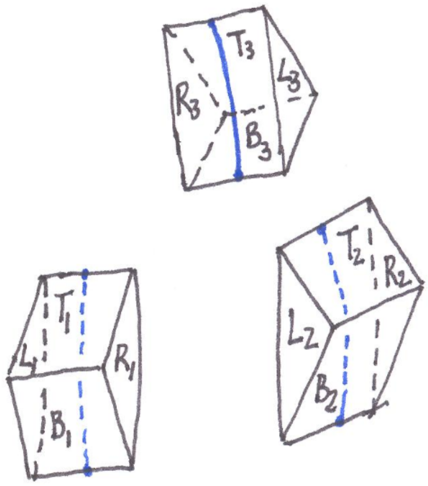
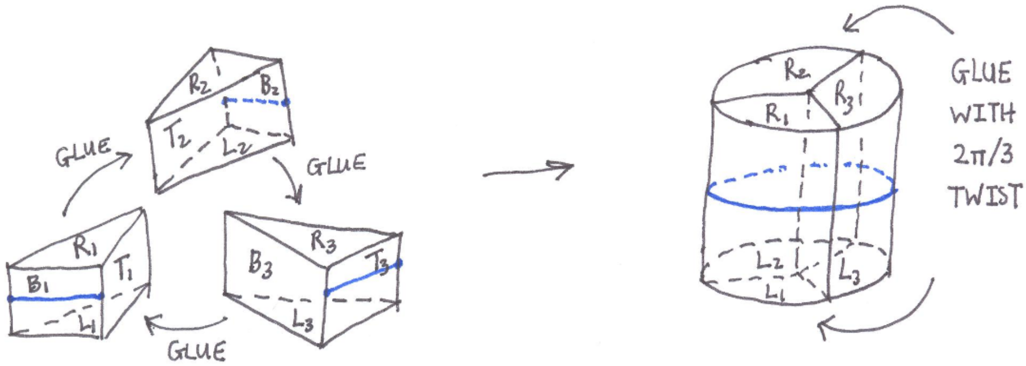


FIGURE 24. Cutting the region V_2



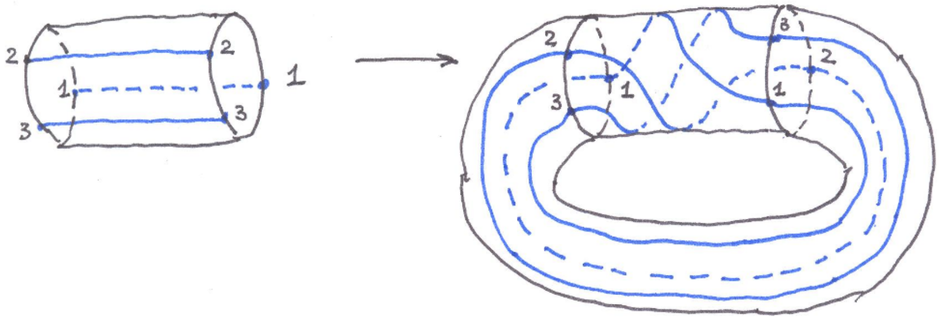
Now instead of first gluing R_i to L_{i+1} and then gluing T_i to B_{i+q}, we first glue T_i to B_{i+q} and then glue R_i to L_{i+1}. These give the same space V_2 under identification. The gluing is carried out as in Figure 25. We then obtain a cylinder, with the top glued to

FIGURE 25. Constructing the solid torus T_2



the bottom with a twist. This gives us the second solid torus T_2. Notice that a meridian of T_2 corresponds to p equally-spaced curves on the boundary of the cylinder V_1. Under the identification, we obtain a curve on the boundary of T_1 that runs p times along a longitude and q times along a meridian. A Heegaard diagram of $L(p, q)$ is therefore given as in Figure 26.

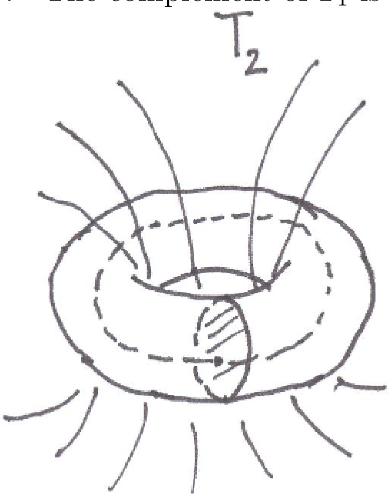
FIGURE 26. A Heegaard diagram of $L(3,2)$ (figure on the right)



3. Surgery Description of 3-Manifolds

We've seen in the previous section that the lens space $L(p,q)$ can be obtained by gluing two solid tori T_1 and T_2 via a homeomorphism that sends a meridian of T_2 to a curve on ∂T_1 that runs p times along a longitude and q times along a meridian. Recall that $\mathbb{S}^3$ can be obtained by gluing two solid tori T_1 and T_2 via a homeomorphism that interchanges longitudes and meridians. In other words, if we remove the interior of the solid torus T_1 from $\mathbb{S}^3$, we're left with the solid torus T_2. In Figure 27, the solid torus T_2 is the space outside the boundary torus. Now to obtain $L(p,q)$, we need to glue in a solid torus

FIGURE 27. The complement of T_1 is a solid torus



$T = \mathbb{D}^2 \times \mathbb{S}^1$ via a homeomorphism that sends a meridian of T to a curve on ∂T_2 that runs p times along a longitude and q times along a meridian of T_2, which is a curve that runs p times along a meridian and q times along a longitude of the solid torus T_1. To summarize, we can obtain $L(p,q)$ from $\mathbb{S}^3$ by first removing a solid torus T_1 (in this case a tubular neighborhood of the unknot) and then gluing in another solid torus T via a homeomorphism that sends a meridian of T to a curve on ∂T_1 that runs p times along a meridian and q times along a longitude of T_1. This procedure can be generalized to links in $\mathbb{S}^3$. It is our goal in this section to show that all closed, orientable, connected 3-manifolds arise in this way.

Definition 3 (Linking Numbers)**.** Let J and K be two disjoint oriented knots in $\mathbb{S}^3$. Consider a regular projection of $J \cup K$. At each point at which J crosses under K, assign an integer ± 1 as shown in Figure 28. The sum of these integers over all crossings of J

under K is called the linking number of J and K, denoted by $\mathrm{lk}(J, K)$. Note that we have $\mathrm{lk}(J, K) = \mathrm{lk}(K, J)$.

FIGURE 28. Positive and negative crossings

Definition 4 (Dehn surgery on $\mathbb{S}^3$)**.** Given a link $L = L_1 \cup \cdots \cup L_k$ in $\mathbb{S}^3$. For each component L_i of L we attach a rational number $r_i \in \mathbb{Q} \cup \{\infty\}$. We'll describe how to obtain a 3-manifold M from these data. First we write each r_i as a ratio of two integers as in the following convention:

(1) if $0 < r_i < \infty$, then $r_i = p_i/q_i$, where p_i and q_i are positive integers which are relatively prime,
(2) if $r_i < 0$, then $r_i = (-p_i)/q_i$, where p_i and q_i are positive integers which are relatively prime,
(3) if $r_i = 0$, then $r_i = 0/1$,
(4) if $r_i = \infty$, then $r_i = 1/0$.

Now we arbitrarily orient each component L_i of L. Let N_i's denote disjoint closed tubular neighborhoods of L_i's in M. On ∂N_i, choose a *preferred longitude* λ_i, i.e. a simple closed curve whose linking number with L_i is 0, oriented in the same way as L_i, and a meridian μ_i with an orientation such that $\mathrm{lk}(\mu_i, L_i) = +1$. Let J_i be a curve on ∂N_i given by

$$J_i = p_i\mu_i + q_i\lambda_i.$$

Since p and q are relatively prime, J_i is a simple closed curve on ∂N_i. Now we construct the 3-manifold

$$M = (\mathbb{S}^3 - (\mathring{N}_1 \cup \cdots \cup \mathring{N}_k)) \cup_h (N_1 \cup \cdots \cup N_k),$$

where h is a union of homeomorphisms $h_i \colon \partial N_i \to \partial N_i \subset \mathbb{S}^3$, each of which takes a meridian μ_i of N_i onto the specified J_i. Note that the choice of orientations of J_i is irrelevant because if we reverse the orientation of L_i, then we also reverse the orientations of λ_i and μ_i and so

$$\widetilde{J_i} = -p_i\mu_i - q\lambda_i,$$

which is isotopic to the original J_i. The 3-manifold M is said to be the result of a *Dehn surgery* on $\mathbb{S}^3$ along the link L with *surgery coefficients* r_i.

Remark 2. 1. Note that

$$\mathrm{lk}(L_i, J_i) = \mathrm{lk}(L_i, p_i\mu_i + q_i\lambda_i) = p_i\mathrm{lk}(L_i, \mu_i) + q_i\mathrm{lk}(L_i, \lambda_i) = p_i$$

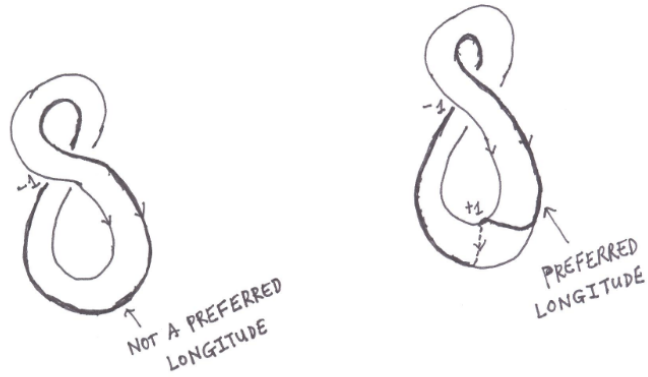
and

$$\mathrm{lk}(\mu_i, J_i) = \mathrm{lk}(\mu_i, p_i\mu_i + q_i\lambda_i) = p_i\mathrm{lk}(\mu_i, \mu_i) + q_i\mathrm{lk}(\mu_i, \lambda_i) = q_i$$

because $\mathrm{lk}(L_i, \mu_i) = 1$ and $\mathrm{lk}(L_i, \lambda_i) = 0$ and $\mathrm{lk}(\mu_i, \mu_i) = 0$.

2. The choice of λ_i may not be the "obvious" one. (See Figure 29)

FIGURE 29. Preferred longitude



Definition 5 (Dehn twist)**.** Let F be closed oriented surface. Let C be a simple closed curve embedded in F, and let A be an annulus neighborhood of C. Identify A with the region $1 \leq |z| \leq 2$ in the complex plane $\mathbb{C}$ (this region inherits the natural orientation of $\mathbb{C}$) in an orientation preserving manner. A *Dehn twist* about C is any homeomorphism isotopic to the homeomorphism $\tau \colon F \to F$ defined such that τ is the identity on $F - A$ and τ is given on A by $\tau(re^{i\theta}) = re^{i(\theta - 2\pi(r-2))}$ (see Figure 30 and Figure 31).

FIGURE 30. The Dehn twist τ_C (top view)

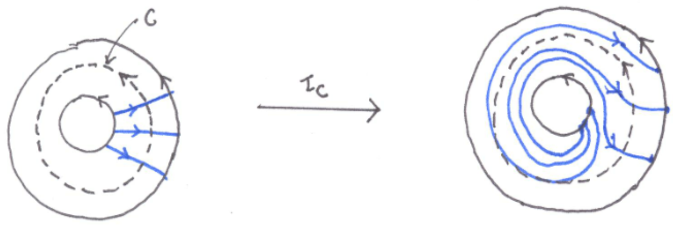
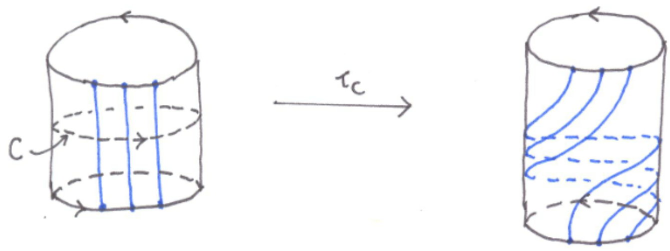


FIGURE 31. The Dehn twist τ_C (side view)



Example 5. 1. Figure 32 shows a composition of two Dehn twists that interchange a meridian and a longitude of a torus.

2. We can similarly define negative Dehn twist. Figure 33 shows a composition of positive Dehn twist and negative Dehn twist give the identity.

Theorem 2 (The Lickorish Twist Theorem)**.** *Let F be a closed orientable surface of genus g. Then every orientation-preserving homeomorphism of F is isotopic to the identity or to a product of Dehn twists along the $3g-1$ curves pictured in Figure 34.*

Proof. We refer the readers to [9] for a beautiful proof of the above theorem. □

Figure 32. Interchange the meridian and longitude of a torus

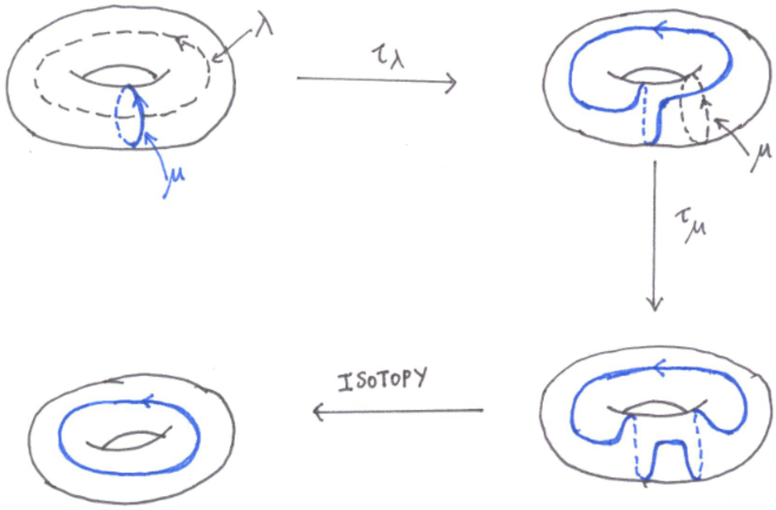


Figure 33. Cancellation of positive Dehn twist and negative Dehn twist

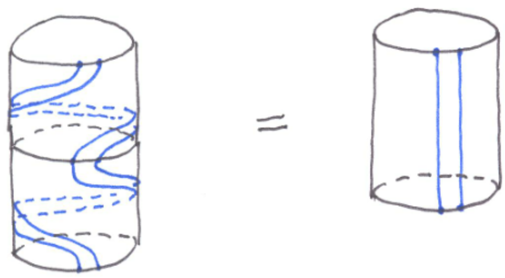
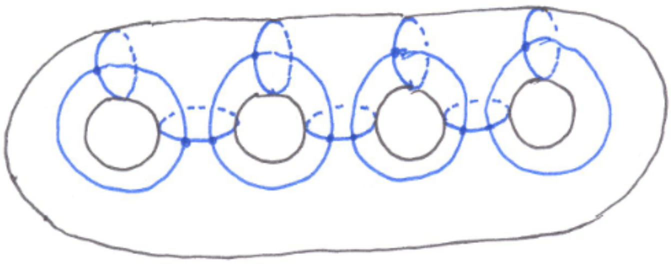


Figure 34. The $3g-1$ generating curves

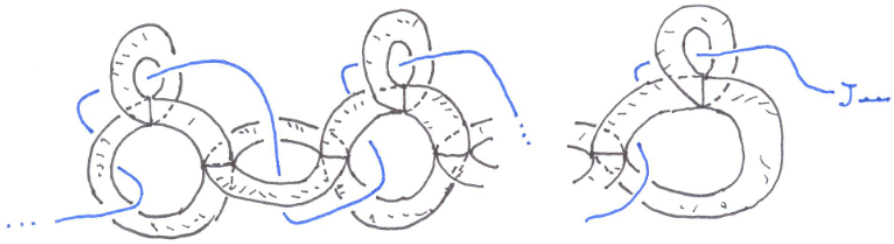


Theorem 3. *Every closed, oriented, connected 3-manifold may be obtained by surgery on a link L in $\mathbb{S}^3$. Each component of L is unknotted and has surgery coefficient ± 1. Moreover, the link L can be chosen to lie in the region pictured in Figure 35*

Proof. Since M is closed, oriented and connected, it has a Heegaard splitting of genus g (Proposition 3) $M = H_1 \cup_h H_2$. From Example 1 part (3), we know that $\mathbb{S}^3$ has a Heegaard splitting of the same genus $\mathbb{S}^3 = H_1 \cup_{h'} H_2$. We may choose H_2 to be standardly embedded in $\mathbb{S}^3$ and H_1 the closure of its complement. The proof proceeds in two steps as follows.

Step 1: We show that if $f\colon \partial H_2 \to \partial H_2$ is an orientation-preserving homeomorphism, then there exist disjoint solid tori $V_1, ..., V_r$ in H_2 and $V_1', ..., V_r'$ in H_2 such that f extends

Figure 35. The region that contains the surgery link L in $\mathbb{S}^3$

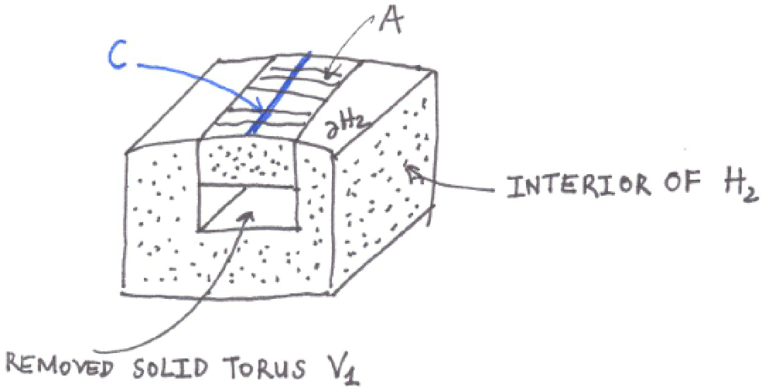


to a homeomorphism

$$\widetilde{f}\colon H_2 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to H_2 - (\mathring{V}'_1 \cup \cdots \cup \mathring{V}'_r).$$

From Theorem 2, the homeomorphism f is either isotopic to the identity or to a composition of Dehn twists. If f is isotopic to the identity, then we can think of the isotopy as happening in a collar neighborhood of ∂H_2 and so we can extend f to be the identity off the collar neighborhood of ∂H_2. Now suppose f is isotopic to a composition of Dehn twist along some of the $3g-1$ canonical curves (see Figure 34), say $f = \tau_r \cdots \tau_2\tau_1$ (note that some of the τ_i's may be twists along the same curve). Suppose τ_1 is the twist along some curve C on ∂H_2. Let A be an annular neighborhood of C in ∂H_2. Recall that τ_1 is the identity off A. Remove a solid torus, call it V_1 from H_2 just under A and within a collar neighborhood of ∂H_2 in H_2 (see Figure 36). The region between V_1 and A is a copy

Figure 36. Excavating a solid torus



of $A \times [0,1]$, which may be twisted by $\tau_1 \times \mathrm{Id}$ (see Figure 37). This map, which extends

Figure 37. Twisting $A \times [0,1]$

τ_1, can be extended as the identity to the rest of $H_2 - \mathring{V}_1$. We thus obtain the extension

$$\widetilde{\tau}_1\colon H_2 - \mathring{V}_1 \to H_2 - \mathring{V}_1.$$

Similar, we obtain the extensions

$$\widetilde{\tau}_i \colon H_2 - \mathring{V}_i \to H_2 - \mathring{V}_i, \quad i = 2, ..., r.$$

Note that we can choose the solid tori $V_1, ..., V_r$ so that $\widetilde{\tau}_i$ is fixed on V_j whenever $i < j$. Moreover, if τ_i and τ_j are twists along the same curve C with $i < j$, we can isotope C slightly to obtain C' disjoint from C, and choose solid tori V_i and V_j underneath annular neighborhoods A and A' of C and C' respectively, so that V_i and V_j are disjoint and $A \times [0,1]$ and $A' \times [0,1]$ are disjoint. Now we put $\widetilde{f} = \widetilde{\tau}_r \cdots \widetilde{\tau}_1$. To have a well-defined composition, we let the domain of $\widetilde{f}$ be $H_2 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r)$. Then the range of $\widetilde{f}$ is given by $H_2 - (\mathring{V}'_1 \cup \cdots \cup \mathring{V}'_r)$, where

$$V'_r = V_r \text{ and } V'_i = \tau_r \cdots \tau_{i+1}(V_i) \text{ for } i < r.$$

(This follows from the condition that $\widetilde{\tau}_i(V_j) = V_j$ for $i < j$.) Note also that the V'_i are disjoint solid tori. We've obtained an extension $\widetilde{f}$.

Step 2: From Step 1, the homeomorphism $h^{-1}h' \colon \partial H_2 \to \partial H_2$ can be extended to a homeomorphism

$$F \colon H_2 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to H_2 - (\mathring{V}'_1 \cup \cdots \cup \mathring{V}'_r).$$

This homeomorphism can be extended to

$$F \colon \mathbb{S}^3 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to M - (\mathring{V}'_1 \cup \cdots \cup \mathring{V}'_r)$$

by defining F to be the identity on H_1. Let L be the union of the cores of V_i's, we obtain a link L in $\mathbb{S}^3$ whose each component is the unknot. The homeomorphism H tells us that if we remove solid tori $V'_i s$ from $\mathbb{S}^3$, we obtain a homeomorphic copy of $M - (\mathring{V}'_1 \cup \cdots \cup \mathring{V}'_r)$. To obtain M therefore, we need to glue in solid tori V'_i via homeomorphisms that send meridians of V'_i to meridians of V'_i. To recover the surgery coefficients of L, we need to find the preimages of the meridians of V'_i. For that, we look at the image of the meridians of V_i under F. From Step 1, we observe that the extension $\widetilde{\tau}_i$ sends a meridian of V_i to a meridian $\pm$ a longitude of V_i. If C' is a curve which intersects C transversally, twist along C' will have no effect on the meridian. Therefore the preimage of a meridian of V'_i is a meridian $\pm$ longitude of V_i. It follows that the surgery coefficients of each component of L is ± 1. ☐

Example 6. 1. We've seen in the beginning of this section that the lens space $L(p, q)$ is the result of a surgery on $\mathbb{S}^3$ along the unknot with surgery coefficient p/q.

2. The sphere $\mathbb{S}^3$ is obtained by removing a tubular neighborhood T_1 of the unknot and then gluing in a solid torus via a homeomorphism that sends a meridian to a meridian on the boundary of T_1. Therefore, the surgery coefficient in this case is ∞. In fact, it turns out that a surgery on $\mathbb{S}^3$ along the unknot with surgery coefficient $1/n$ also gives us $\mathbb{S}^3$. This follows since a surgery with coefficient $1/n$ is $L(1, n)$ and we know that $L(1, n)$ is just $\mathbb{S}^3$ because the identification on $\mathbb{S}^3$ is given by

$$(z, w) \equiv (\exp(2\pi i)z, \exp(2\pi n i)w) = (z, w),$$

which is just the identity.

3. Recall that $\mathbb{S}^1 \times \mathbb{S}^2$ is obtained by gluing two solid tori via a homeomorphism that sends a meridian to a meridian (Example 3). Therefore to get $\mathbb{S}^1 \times \mathbb{S}^2$ we remove a tubular neighborhood of the unknot in $\mathbb{S}^3$ and glue in a solid torus via a homeomorphism that sends a meridian to a longitude of the removed solid torus (which is a meridian of the complementary solid torus). Hence $\mathbb{S}^1 \times \mathbb{S}^2$ is the result of a surgery on $\mathbb{S}^3$ along the unknot with surgery coefficient 0.

Remark 3. One corollary of Theorem 3 is that every closed, oriented, connected 3-manifold is the boundary of a 4-manifold. This is done via an operation on manifolds known as attaching handles. Given an n-manifold N, a λ-handle is a space homeomorphic to $\mathbb{D}^\lambda \times \mathbb{D}^{n-\lambda}$ (note that we need to take into account the dimension of the manifold). The boundary of a λ-handle consists of two parts

$$\mathbb{S}^{\lambda-1} \times \mathbb{D}^{n-\lambda} \cup \mathbb{D}^\lambda \times \mathbb{S}^{n-\lambda-1}.$$

Notice that these two parts have the same boundary:

$$\mathbb{S}^{\lambda-1} \times \mathbb{S}^{n-\lambda-1}.$$

Then the manifold

$$N' = N \cup_h \mathbb{D}^\lambda \times \mathbb{D}^{n-\lambda},$$

where $h\colon \mathbb{S}^{\lambda-1} \times \mathbb{D}^{n-\lambda} \to \partial N$ is an embedding, is called the result of N by *attaching a λ-handle*. Note that the boundary of N' is given by

$$\partial N' = (\partial N - \mathbb{S}^{\lambda-1} \times \mathbb{D}^{n-\lambda}) \cup_k \mathbb{D}^\lambda \times \mathbb{S}^{n-\lambda-1},$$

where the gluing map k is the restriction of h to $\mathbb{S}^{\lambda-1} \times \mathbb{S}^{n-\lambda-1}$. Now back to our case, let $N = \mathbb{D}^4$, the unit 4-disk and let M is any closed, oriented, connected 3-manifold. In this case, a 2-handle is $\mathbb{D}^2 \times \mathbb{D}^2$. The result of attaching a 2-handle to $\mathbb{D}^4$ is the manifold

$$N' = \mathbb{D}^4 \cup_h \mathbb{D}^2 \times \mathbb{D}^2$$

where $h\colon \mathbb{S}^1 \times \mathbb{D}^2 \to \partial \mathbb{D}^4 = \mathbb{S}^3$ is an embedding. The boundary of N' is

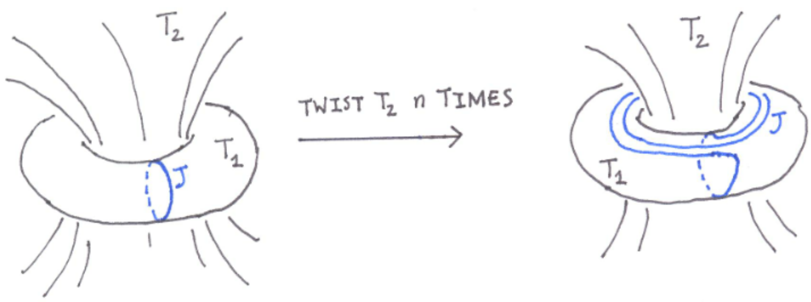
$$\partial N' = (\mathbb{S}^3 - \mathbb{S}^1 \times \mathbb{D}^2) \cup_k \mathbb{D}^2 \times \mathbb{S}^1,$$

where k is the restriction of h to $\mathbb{S}^1 \times \mathbb{S}^1$. We see that $\partial N'$ is precisely the result of a surgery of $\mathbb{S}^3$ along a knot. Theorem 3 now tells us that we can successively attach 2-handles to $\mathbb{D}^4$ to obtain a manifold $\widetilde{N}$ whose boundary is the given manifold M.

4. Modifications of Surgery Coefficients

Let T_1 be a tubular neighborhood of the unknot in $\mathbb{S}^3$. The sphere $\mathbb{S}^3$ is obtained by surgery along the unknot with surgery coefficient ∞, i.e. the curve J runs once along the meridian of T_1 and zero times along the longitude of T_1. The complement T_2 of T_1 in $\mathbb{S}^3$ is a solid torus. Hence we can get a homeomorphic copy of T_2 by performing n Dehn twists along the meridian of T_2, or longitude of T_1 ($n < 0$ corresponds to negative Dehn twist), see Figure 38. This homeomorphism changes J to a curve that runs once along

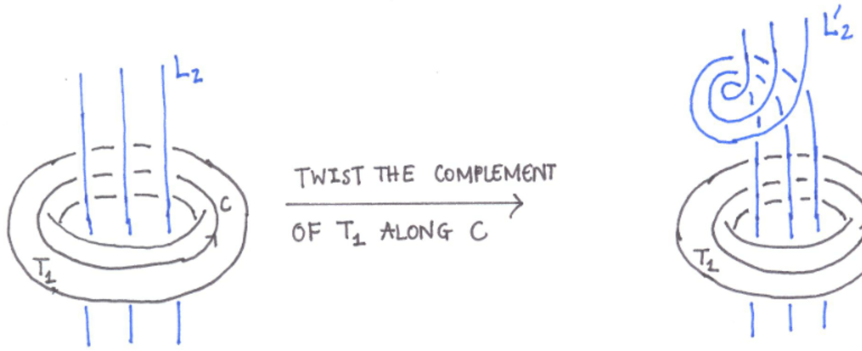
Figure 38. Twisting the solid torus T_2



the meridian and n times along the longitude of T_1. The surgery coefficient is then $1/n$. This gives another reason why $\mathbb{S}^3$ is obtained by surgery along the unknot with surgery coefficient $1/n$. This example illustrates that we can obtain different surgery descriptions of a manifold by fiddling with the complement of the unknot. We'll explore this idea in more details in this section.

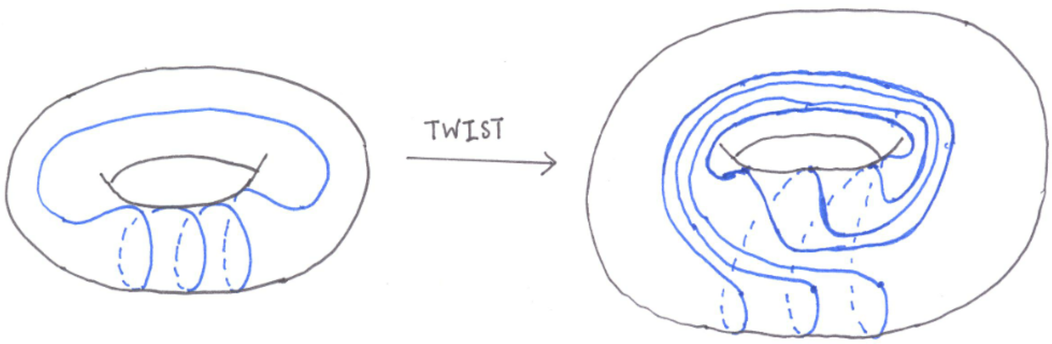
Given a link $L = L_1 \cup \cdots \cup L_n$ in $\mathbb{S}^3$ and suppose L_1 is the unknot. Let T_i be a tubular neighborhood of L_i. The complement of T_1 is a solid torus, which we can revise with a meridional twist. This alters the rest of the link as in Figure 39.

FIGURE 39. Twisting of other link components



The new link $L' = L_1 \cup L_2' \cup \cdots \cup L_n'$ gives the same surgery manifold. However we need to revise the surgery coefficients. First look at $r_1 = p_1/q_1$. The twist sends J_1 to a curve that runs p_1 times along the meridian and $q_1 + p_1$ times along the longitude (see Figure 40). Therefore the new surgery coefficient r_1' is

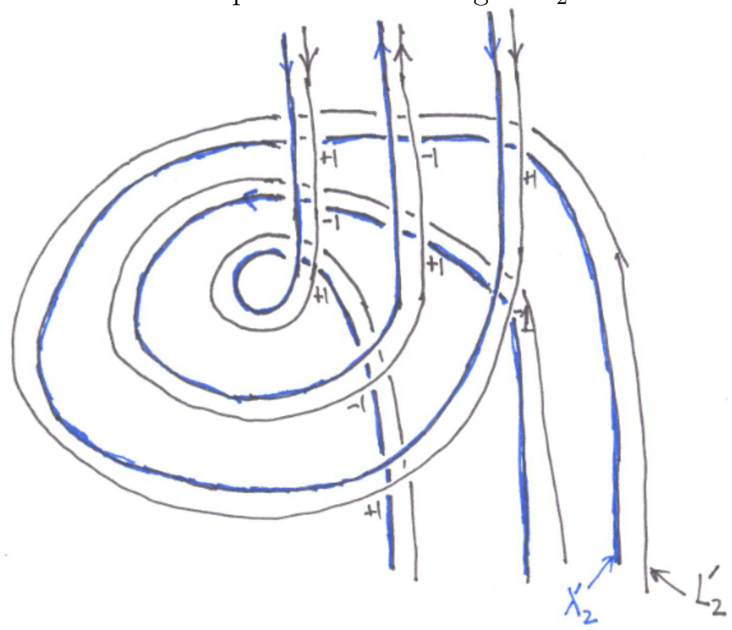
FIGURE 40. Twisting of the curve J_1



$$r_1' = \frac{p_1}{p_1 + q_1} = \frac{1}{1 + \dfrac{q_1}{p_1}} = \frac{1}{1 + \dfrac{1}{r_1}}.$$

Now to find the surgery coefficient r_2' of L_2', we let D denote the disk bounded by L_1. Suppose we have u bits of L_2 passing through D in an upward direction and d bits of L_2 passing through D in a downward direction. Here we represent each string of L_2 as a band with two boundary lines. The bold line represents the longitude λ_2 and the other line represents L_2. Note that these two lines are directed in the same way. The twist will alter L_2 as in Figure 41. Under the twist, a meridian of L_2 will be sent to a meridian of L_2' and λ_2 will be sent to λ_2'. However, λ_2' may not be a longitude of L_2' since the linking number of λ_2' with L_2' may not be 0. We'll compute $\mathrm{lk}(\lambda_2', L_2')$ below.

FIGURE 41. Represent each string of L_2 as a band



Consider an upward band, and look at the linking number of the part of λ_2' (the bold line) in the band with L_2'. The linking number of λ_2' with L_2' in the band itself is $+1$. Any other upward band will contribute $+1$ to the linking number and any downward band will contribute -1 to the linking number (see Figure 41). Therefore, for any upward band, the linking number of the part of λ_2' in the band with L_2' is $u-d$. Similarly, for any downward band, the linking number of the part of λ_2' in the band with L_2' is $d-u$. The linking number of λ_2' with L_2' outside the twisted region is 0 since the linking number of λ_2 with L_2 is 0. Therefore, we have

$$\mathrm{lk}(\lambda_2', L_2') = u(u-d) + d(d-u) = (u-d)^2 = [\mathrm{lk}(L_2, L_1)]^2.$$

The curve J_2 is sent to J_2' given by

$$J_2' = p_2\mu_2' + q_2\lambda_2'.$$

Notice that μ_2' is still a meridian of L_2'. To find the revised surgery coefficient r_2' (see Remark 2), we compute

$$p_2' = \mathrm{lk}(L_2', J_2') = \mathrm{lk}(L_2', p_2\mu_2' + q_2\lambda_2') = p_2 + q_2[\mathrm{lk}(L_2, L_1)]^2,$$

and

$$q_2' = \mathrm{lk}(J_2', \mu_2') = \mathrm{lk}(p_2\mu_2' + q_2\lambda_2', \mu_2') = q_2.$$

Therefore

$$r_2' = \frac{p_2'}{q_2'} = \frac{p_2 + q_2[\mathrm{lk}(L_2, L_1)]^2}{q_2} = r_2 + [\mathrm{lk}(L_2, L_1)]^2.$$

By a similar argument, we obtain the following proposition:

Proposition 5. *The surgery coefficients after performing t Dehn twists ($t < 0$ for negative Dehn twist) about an unknotted component L_i of a link L is given by*

$$r_i' = \frac{1}{t + \frac{1}{r_i}},$$

and

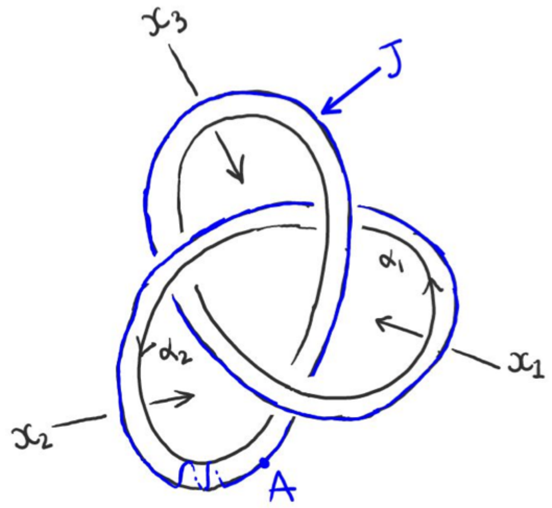
$$r'_j = r_j + t[\mathrm{lk}(L_i, L_j)]^2 \quad \textit{for } j \neq i.$$

Example 7 (Poincare Homology Sphere)**.** The Poincare homology sphere is the 3-manifold obtained from $\mathbb{S}^3$ by surgery along the (right-handed) trefoil with surgery coefficient 1, i.e.

$$P = (\mathbb{S}^3 - \mathring{N}) \cup_h (\mathbb{S}^1 \times \mathbb{D}^2),$$

where N is a closed tubular neighborhood of the trefoil in $\mathbb{S}^3$ and h is a homeomorphism $\mathbb{S}^1 \times \mathbb{D}^2 \to N$ that sends a meridian of $\mathbb{S}^1 \times \mathbb{D}^2$ to a curve J on ∂N that has linking number $+1$ with the trefoil (see Figure 42). To see that P is not $\mathbb{S}^3$, we compute its

Figure 42. The homology sphere



fundamental group. Let X denote the complement of the trefoil in $\mathbb{S}^3$. We first compute the fundamental group of X using the Wirtinger method. We describe the algorithm in this particular case, the reader can consult [12] for more details.

Consider the projection of the trefoil in the xy-plane. We first orient the trefoil and divide the trefoil into oriented arcs α_1, α_2, α_3. Each α_i is connected to α_{i+1} and α_{i-1} (mod 3) in a way that every time an arc passes through an undercrossing, it switches to the next arc. Now for each arc α_i, draw a short arrow labelled x_i passing under each α_i in a right-left direction when traversed along the direction of α_i (see Figure 42). Each x_i represents an element in $\pi_1(X)$ as follows: take $(0,0,1)$ as the base point, then x_1 represents the loop that goes from $(0,0,1)$ to the tail of x_1, then along x_1 to the head, then back to $(0,0,1)$. The Wirtinger method asserts that the fundamental group of X is given as follows:

$$\pi_1(X) = \langle x_1, x_2, x_3 \colon x_2x_1 = x_1x_3,\ x_1x_3 = x_3x_2,\ x_3x_2 = x_2x_1 \rangle.$$

The relation $x_2x_1 = x_1x_3$ says that $x_3 = x_1^{-1}x_2x_1$. Substitute it back to the other two relations we obtain

$$\pi_1(X) = \langle x_1, x_2 \colon x_1x_2x_1 = x_2x_1x_2 \rangle.$$

To find the fundamental group of P, we think of the gluing map h as a two step process. First we glue a collar neighborhood of the meridian of $\mathbb{S}^1 \times \mathbb{D}^2$, which makes J the boundary of a 2-disk, and then we glue the remaining part of $\mathbb{S}^1 \times \mathbb{D}^2$, which is a 3-ball. Only the first step affects the fundamental group. Therefore

$$\pi_1(P) = \langle x_1, x_2 \colon x_1x_2x_1 = x_2x_1x_2,\ [J] = 1\rangle.$$

Now we need to express J in terms of x_1 and x_2. Start at a point $A \in J$ (see Figure 42) and traverse along J in the direction of the trefoil. Every time J goes under α_i, we write x_i if J goes in the direction of x_i and x_i^{-1} if J goes in the opposite direction of x_i. Thus in our case J is given by

$$J = x_1x_2x_3x_2^{-2} = x_1x_2(x_1^{-1}x_2x_1)x_2^{-2}.$$

Therefore our extra relation is $x_1x_2x_1^{-1}x_2x_1x_2^{-2} = 1$. Multiply both sides by x_2^3 we obtain $x_1x_2x_1^{-1}x_2x_1x_2 = x_2^3$. Now using the relation $x_2x_1x_2 = x_1x_2x_1$ we get $x_1x_2^2x_1 = x_2^3$. Thus

$$\pi_1(P) = \langle x_1, x_2 \colon x_1x_2x_1 = x_2x_1x_2,\ x_1x_2^2x_1 = x_2^3\rangle.$$

Make a substitution $z = x_1x_2$, then $x_2 = x_1^{-1}z$. The first relation becomes $zx_1 = x_2z = x_1^{-1}z^2$, or $(zx_1)^2 = z^3$. The second relation becomes $x_2x_1z = x_1^3$, or $x_1^{-1}zx_1z = x_1^3$, or $zx_1z = x_1^4$. Using $zx_1 = x_1^{-1}z^2$ we obtain $x_1^{-1}z^2z = x_1^4$, or $z^3 = x_1^5$. So $\pi_1(P)$ has the following presentation

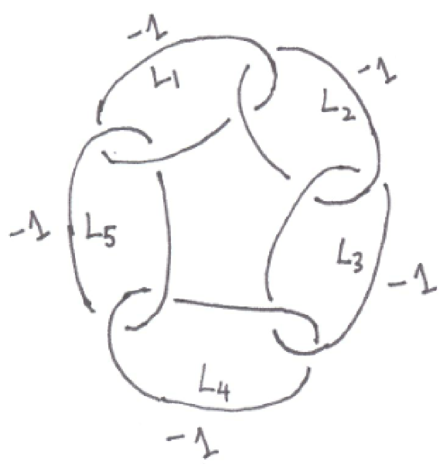
$$\pi_1(P) = \langle x_1, z \colon (zx_1)^2 = z^3 = x_1^5\rangle.$$

To see that $\pi_1(P)$ is non-trivial, we consider the group of rotational symmetries of the regular icosahedron, which has order 60 and is generated by α, a $2\pi/3$ rotation about an axis through centers of opposite faces and β, a $2\pi/5$ rotation about an axis through opposite vertices. If we send $x_1 \mapsto \beta$ and $z \mapsto \alpha$, then it can be checked geometrically that α and β satisfy the relations in the presentation of $\pi_1(P)$. So $\pi_1(P)$ maps homomorphically onto a group of order 60 and is therefore non-trivial.

Note that when $x_1z = zx_1$, then $(zx_1)^2 = z^3$ implies $x_1^2 = z$ and $z^3 = x_1^5$ implies $x_1 = z$. So $x_1 = x_1^2$ and $x_1 = z = 1$. Thus $H_1(P)$, which is the abelianization of $\pi_1(P)$, is trivial. By the Universal Coefficient Theorem, $H^1(P)$ is isomorphic to $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$. Since P is a closed oriented 3-manifold, $H_2(P)$ is isomorphic to $H^1(P) = 0$, by Poincare duality, and $H_3(P) = \mathbb{Z}$. Therefore the homology of P is the same as $\mathbb{S}^3$, hence the name homology sphere.

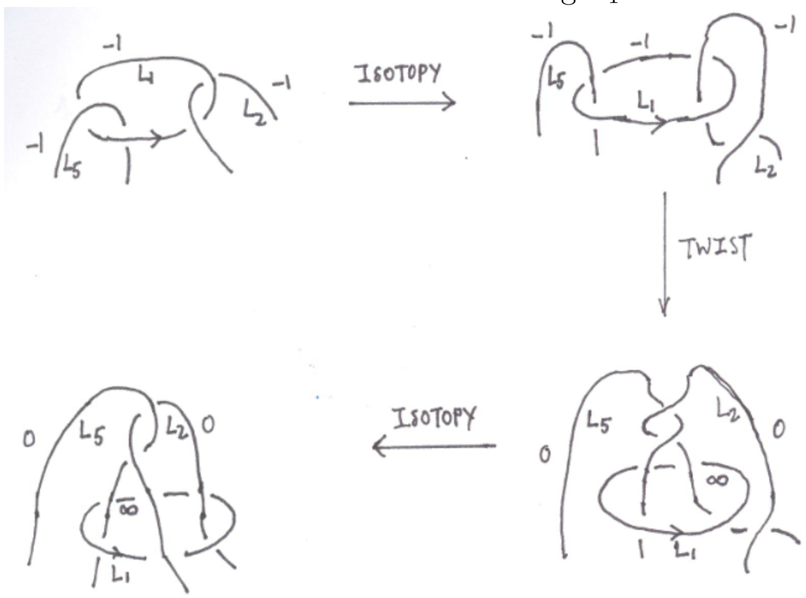
Using modification of surgery, we'll show that the homology sphere P has a surgery description given in Figure 43, where the components of the link are labelled by L_i's. First we perform a Dehn twist along L_1 (Figure 44) (notice how the surgery coefficients

FIGURE 43. A surgery description of the homology sphere



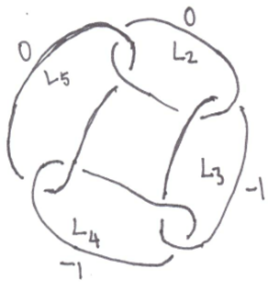
are modified according to Proposition 5). Since surgery along the unknot with coefficient

FIGURE 44. Twist along L_1



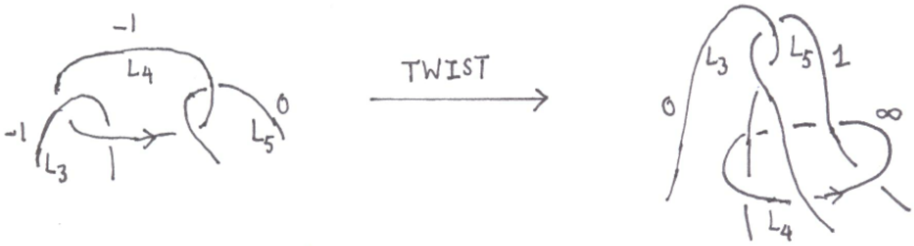
∞ does not change the manifold (it corresponds to removing a solid torus and gluing it back in via the identity), we obtain an equivalent surgery description as given in Figure 45. Similarly, we perform a Dehn twist along L_4 (Figure 46) to obtain Figure 47 (after

FIGURE 45. An equivalent surgery description



removing the infinity component). Next, we perform an inverse (left-hand) Dehn twist

FIGURE 46. Twist along L_4



along L_5 (Figure 48) to obtain the link in Figure 49. Finally, a Dehn twist along L_3

FIGURE 47. An equivalent surgery description

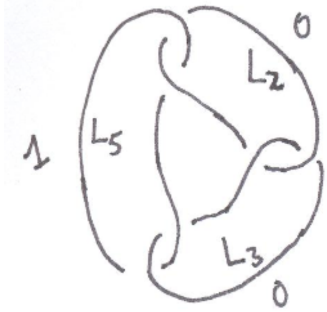


FIGURE 48. Twist along L_5

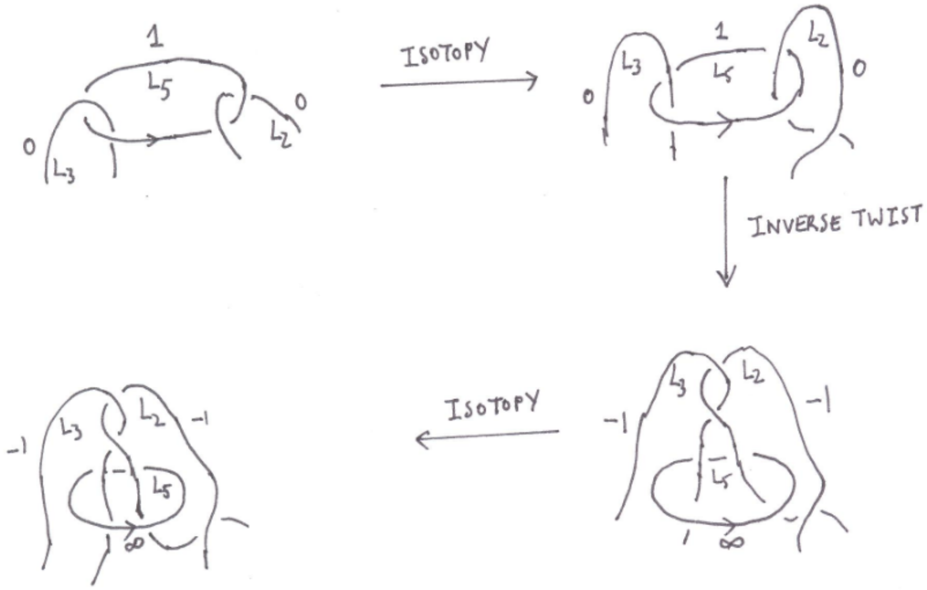


FIGURE 49. An equivalent surgery description

gives us the link as in Figure 50(note that the surgery coefficient is still -1 because $\mathrm{lk}(L_2, L_3) = 0$), which is just the left-hand trefoil. A reflection in $\mathbb{R}^3$ turns the left-hand trefoil into the right-hand trefoil the surgery coefficient from -1 to 1 (Figure 51). We've obtained the description of the homology sphere given in the beginning of this example.

Figure 50. Twist along L_3

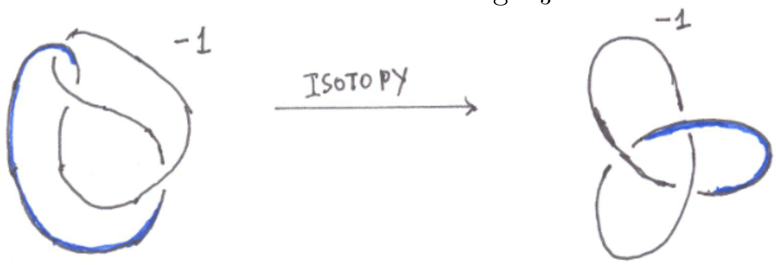
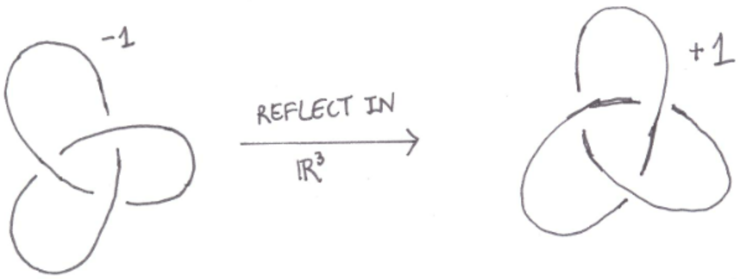


Figure 51. A reflection in $\mathbb{R}^3$



5. Branched Coverings

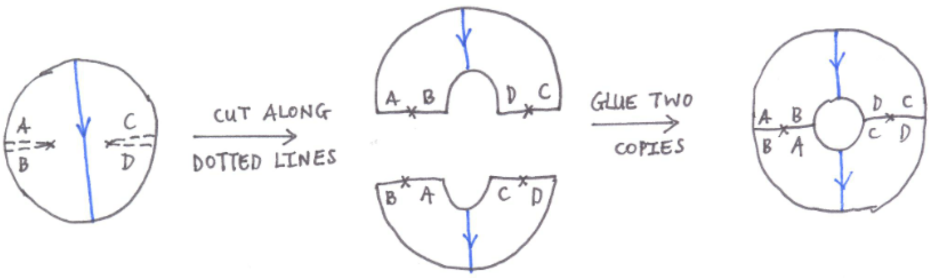
Definition 6. Let M^n and N^n be compact manifolds and $B^{n-2} \subset N$ is an embedded submanifold of codimension 2. Then a map $p\colon M^n \to N^n$ is a *branched covering* along B if $p^{-1}(B)$ is a embedded submanifold of M^n of codimension 2 and $p\colon M - p^{-1}(B) \to N - B$ is a covering space. We call $p^{-1}(B)$ the *branch set upstairs* and B the *branch set downstairs*

Each branch point $a \in p^{-1}(B)$ upstairs has a *branching index* k, meaning that f is k-to-one near a. If $n = 2$, then M and N can be endowed with a complex structure and it can be shown that near each branch point a the map p looks like $z \to z^m$ where $z \in \mathbb{C}$ and m is the branching index of a.

If $p\colon M^n \to N^n$ is a covering and $f\colon N^n \to N^n$ is a homeomorphism, then we say $g\colon M^n \to M^n$ is a *lift* of f if $fp = pg$.

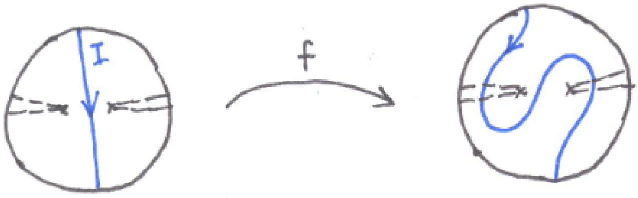
Example 8. We can construct a 2-fold branched cover of a 2-disk along 2 branched points as shown in Figure 52. The covering manifold is the annulus. We have two branch points upstairs, each one has branching index 2.

Figure 52. A 2-fold branched cover of a 2-disk along 2 branched points



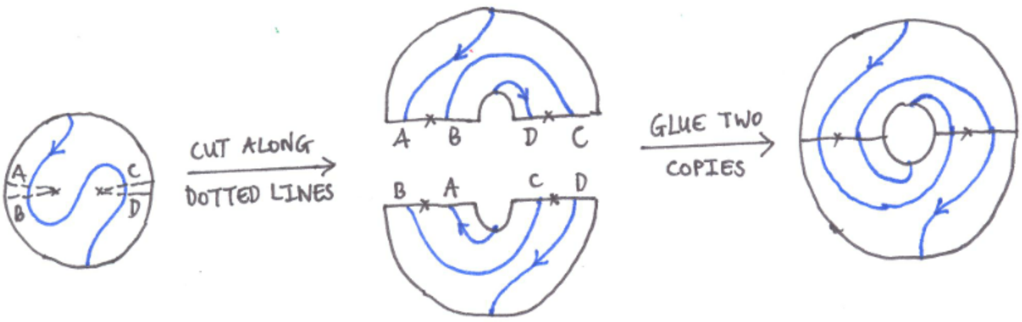
Now suppose that f is a homeomorphism of the disk that is identical on the boundary and interchanges the branch points by a π twist. Figure 53 shows the image of the line segment I under the homeomorphism f. This homeomorphism f lifts to a homeomor-

Figure 53. Interchange two branched points



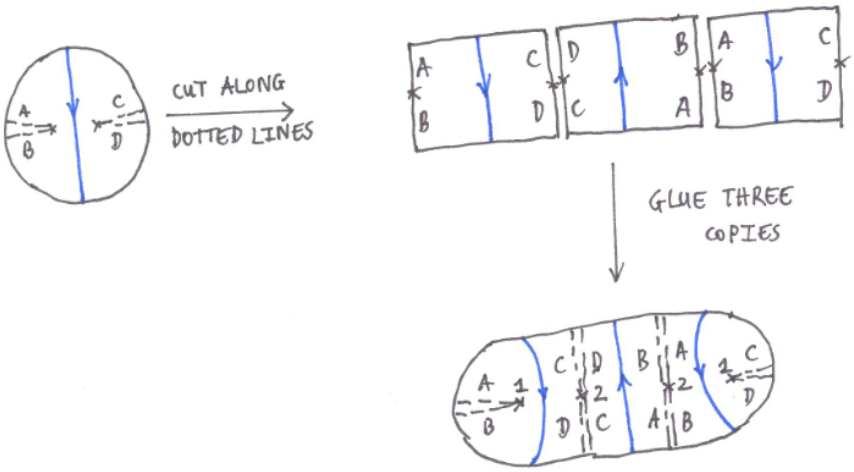
phism of the annuli that is identical on the boundary and twists each branch point by 2π (see Figure 54).

Figure 54. The twist upstairs



Example 9. We can construct a 3-fold branched cover of a 2-disk along 2 branched point as shown in Figure 55. The covering manifold is again a disk. We have four branch points upstairs, two of them have branching index 2 and two of them have branching index 1.

Figure 55. A 3-fold branched cover of a 2-disk along two branched points



Now suppose f is a homeomorphism of the disk, identical on the boundary, interchanging the branch points by a 3π twist. Figure 56 shows the image of the line segment I under the homeomorphism f. Figure 57 shows how to construct a homeomorphism g of

FIGURE 56. The 3π twist map

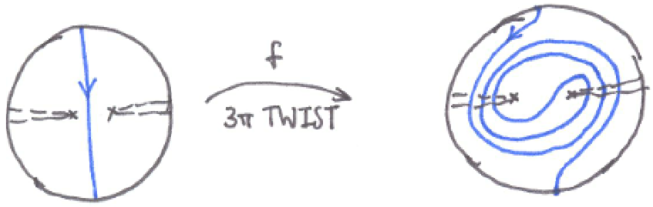
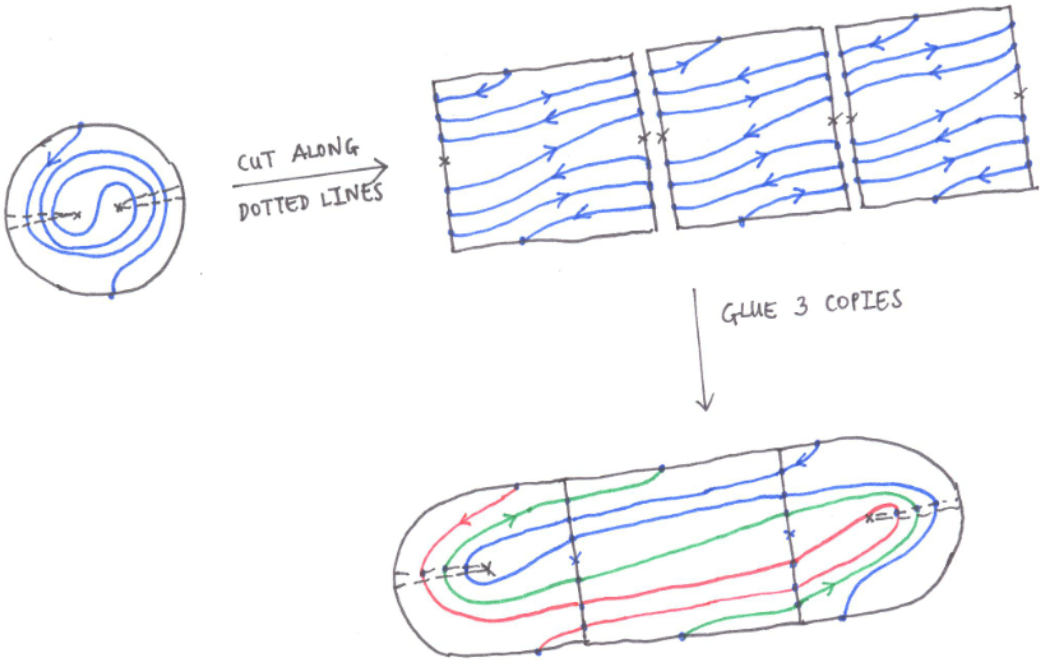


FIGURE 57. The twist upstairs



the disk lifting f that is identical on the boundary. Notice that there is no homeomorphism g of the disk lifting f that is identical on the boundary if we only perform a π or 2π twist to the branch points (see Figure 58 and Figure 59).

Example 10. The 2-fold branched cover of a 2-disk along 3 branch points is a 2-disk with two holes (see Figure 60). The 3-fold branched cover of a 2-disk along 3 branch points is again an annulus. (see Figure 61)

Example 11. An important class of branched covering is the cyclic branched covering of $\mathbb{S}^3$ along the unknot. It is constructed as follows. Think of $\mathbb{S}^3$ as $\mathbb{R}^3 \cup \infty$ and the unknot as some straight line l together with ∞. Now define an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{S}^3$ by a rotation by the angle $2\pi/n$ about the l-axis (note that $\mathbb{Z}/n\mathbb{Z}$ fixes ∞) (see Figure 62). The quotient of $\mathbb{S}^3$ by the action is still $\mathbb{S}^3$. The branched set downstair is the unknot, as is the one upstairs.

Example 12. The above example allows us to construct different branched covers of $\mathbb{S}^3$ branched along the trefoil. First perform surgery of $\mathbb{S}^3$ along the knot J that goes around the l-axis as pictured in Figure 63 with surgery coefficient 1. Notice that since J is the unknot, the result of surgery is still $\mathbb{S}^3$. Now to turn $l \cup \infty$ into the trefoil, we deform J into the standard embedding of the unknot in $\mathbb{S}^3$. Then $l \cup \infty$ will be deformed as in Figure 64. Now we perform a surgery modification about J (see Proposition 5), which

FIGURE 58. The π twist map

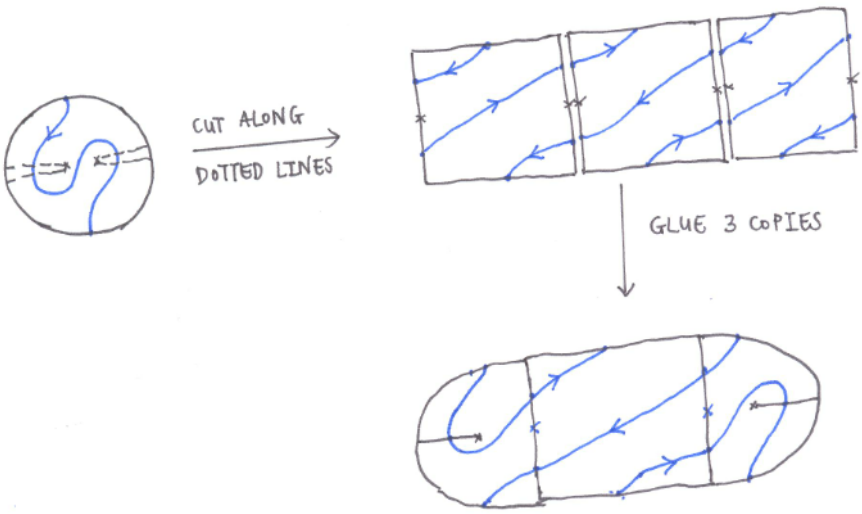


FIGURE 59. The 2π twist map

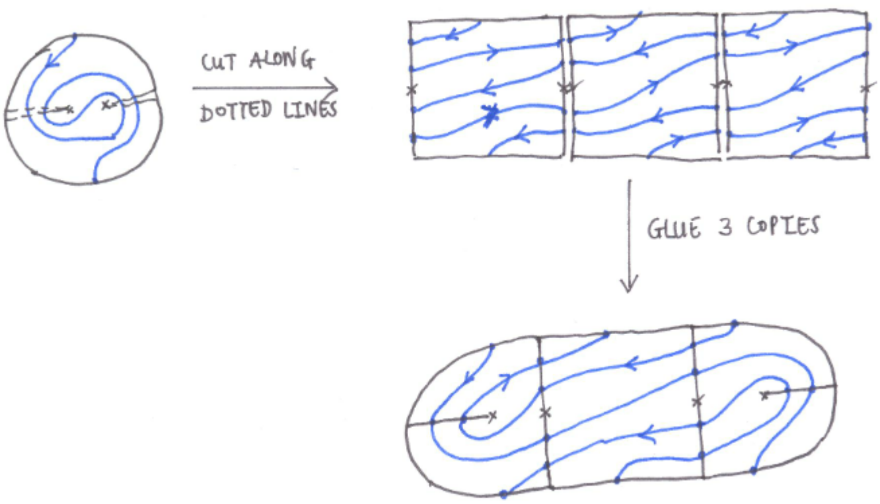
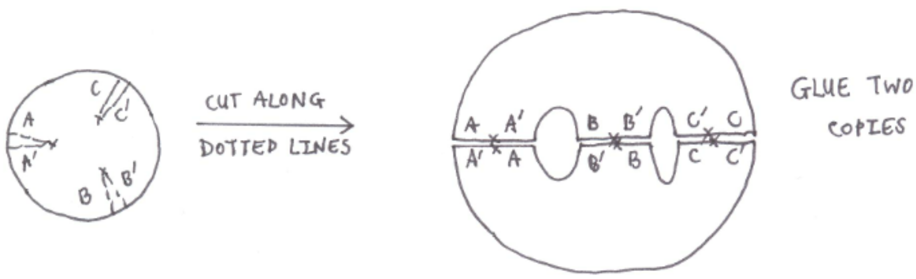


FIGURE 60. The 2-fold branched cover of a 2-disk along three branched points



turns $l \cup \infty$ into the trefoil (see Figure 65).

Now let's see some 3-manifolds that we can get upstairs. We need to look at the preimage of J in the covering $\mathbb{S}^3$. Figure 66 shows the preimage of J in the 2-fold cyclic

FIGURE 61. The 3-fold branched cover of a 2-disk along three branched points

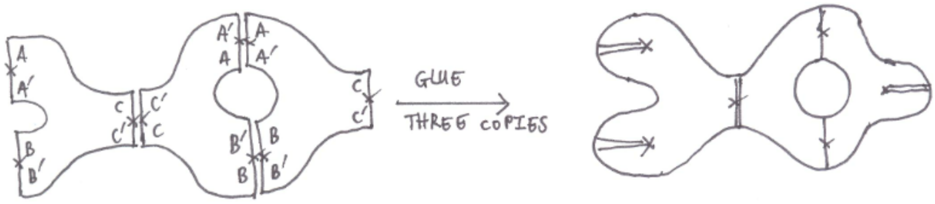
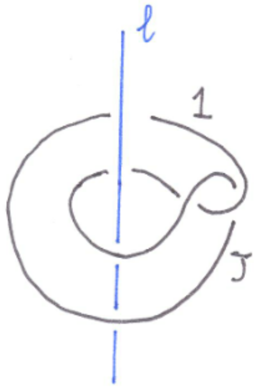


FIGURE 62. A cyclic branched cover of $\mathbb{S}^3$

FIGURE 63. The knot J



covering of $\mathbb{S}^3$, notice how the surgery coefficient changes. To see what manifold we get, apply a twist along L_1 (see Figure 67) followed by an isotopy. The result is the surgery along the unknot with surgery coefficient 3, which is the lens space $L(3,1)$ (recall Example 6). If we consider the 5-fold cyclic covering of $\mathbb{S}^3$, then upstairs we obtain the Poincare homology sphere (see Figure 68).

We notice that in all these examples, the base manifold is $\mathbb{S}^3$. We obtain different 3-manifolds in the branched covering. In fact all closed, connected, orientable 3-manifolds are branched covers of $\mathbb{S}^3$ (although not all come from cyclic branched covering). Our main result of this section is the following:

Theorem 4 (Hilden–Montesinos). *Every closed, connected, orientable 3-manifold is a 3-fold branched covering of* $\mathbb{S}^3$ *along a knot.*

Proof. The main idea of the proof goes as follows. We first start with a particular 3-fold covering $\mathbb{S}^3 \to \mathbb{S}^3$. We then perform surgery to the base and covering manifolds so that the base manifold is still $\mathbb{S}^3$ (this can be achieved by surgery along 3-balls) and the

FIGURE 64. A sequence of isotopies

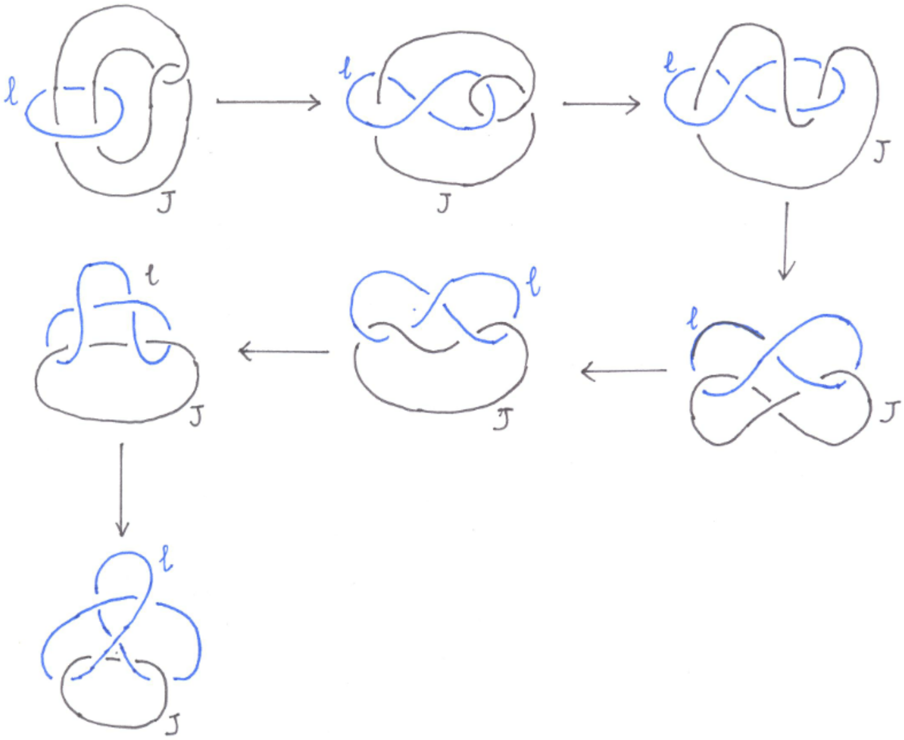
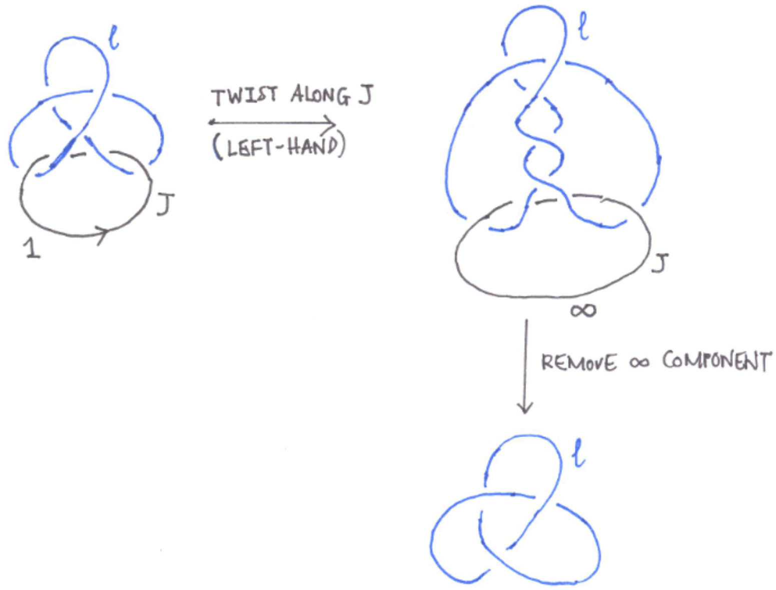


FIGURE 65. Twist along J gives us the trefoil



covering manifold becomes the given 3-manifold M (this is possible by Theorem 3). The details are a bit involved, so we divide the proof into several steps.

Step 1: First we construct a particular 3-fold branched covering $p\colon \mathbb{S}^3 \to \mathbb{S}^3$. Take the Cartesian product of the covering map from Example 9 by the segment $[0,1]$ (see Figure 69, where the bold line segments correspond to points of branching index 2). Both the base and cover manifolds are 3-balls. To obtain a 3-fold branched covering $p\colon \mathbb{S}^3 \to \mathbb{S}^3$, we glue two copies of the base 3-balls to each other via the identity map on the boundary 2-sphere and do the same with two copies of the covering 3-balls.

FIGURE 66. The preimage of J in the 2-fold cover

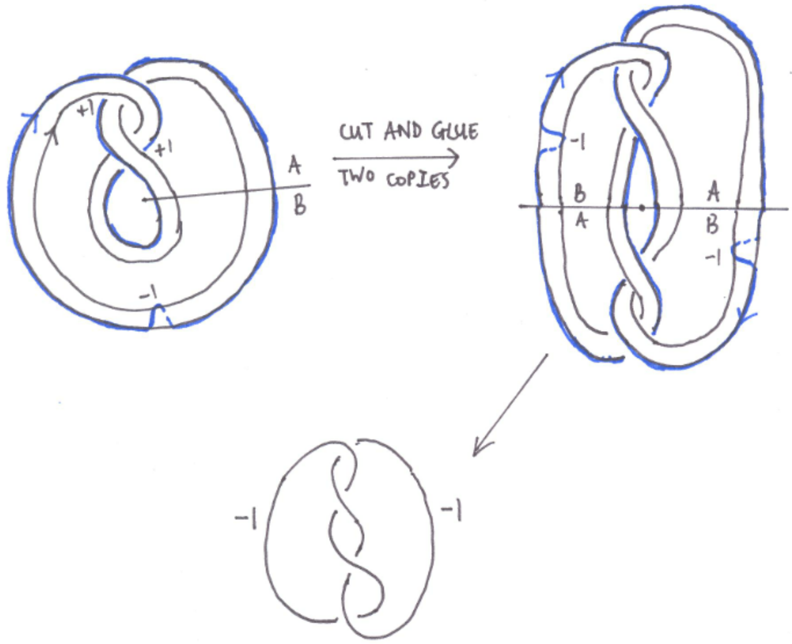


FIGURE 67. Twist along L_1 to obtain the unknot

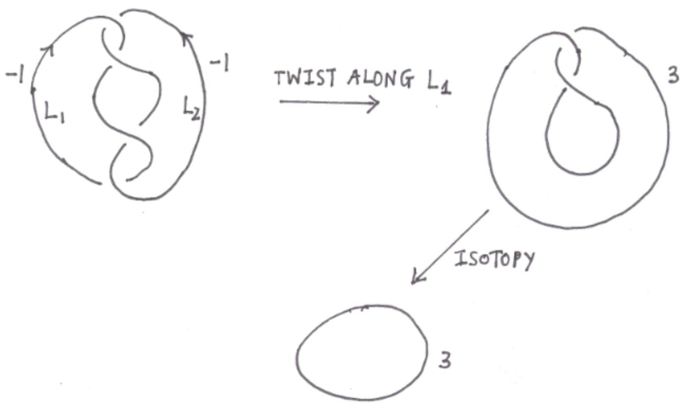
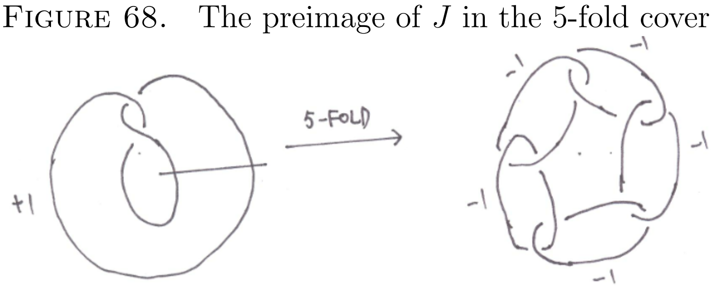


FIGURE 68. The preimage of J in the 5-fold cover



Step 2: In the base $\mathbb{S}^3$ consider a 3-ball D_1^3 intersecting the branch set as the picture on the right in Figure 70. The preimage $p^{-1}(D_1^3)$ in the covering $\mathbb{S}^3$ consists of a 3-ball D_2^3 and a solid torus T_2^3 (see Figure 70, picture on the left). Figure 71 shows how to obtain

FIGURE 69. A 3-fold branched cover of $\mathbb{S}^3$

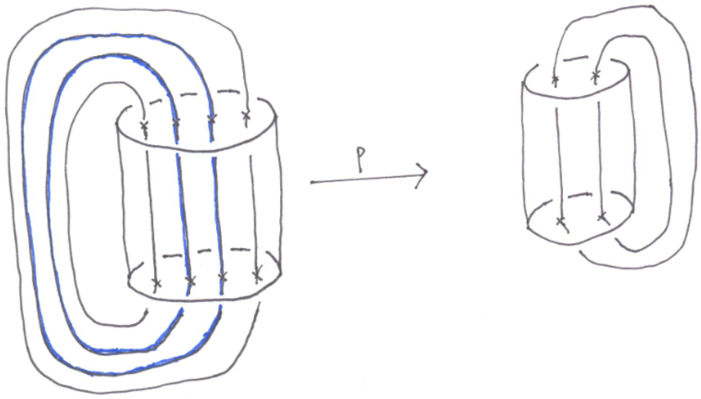
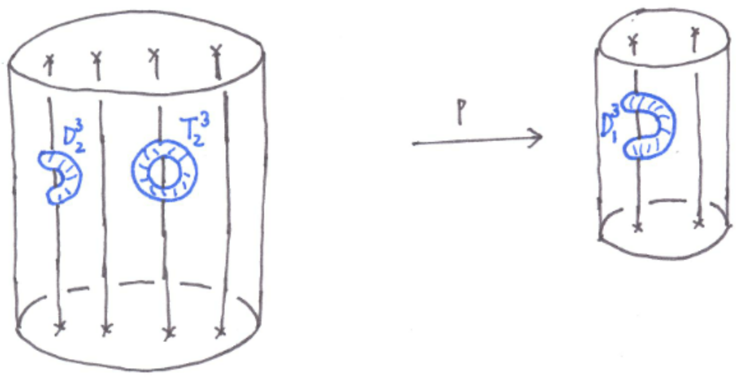


FIGURE 70. The disk D_1^3 and its preimages



the preimage $p^{-1}(D_1^3)$ in $\mathbb{S}^3$, viewed from the top. The restriction of p to D_2^3 is just the

FIGURE 71. The preimage of D_1^3 viewed from the top

identity. A bit of investigation shows that the map $p\colon T_2^3 \to D_1^3$ is just the Cartesian product of the branched covering map from Example 8 by the interval $[0,1]$ (see Figure 72, where we position the 2-disk so that it intersects the branch set at two points and Figure).

Next consider the twist homeomorphism f of the 2-disk from Example 8. Consider the Cartesian product of the 2-disk with $[0,1]$, which is a 3-ball. Since f is identity on the boundary of the 2-disk, we can extend f to the whole boundary 2-sphere. Call the extending map f_1 (see Figure 74). Notice that when we interchange the two branch points, we create a positive crossing or a negative crossing depending on the direction of

FIGURE 72. The branched covering in Example 8 positioned differently

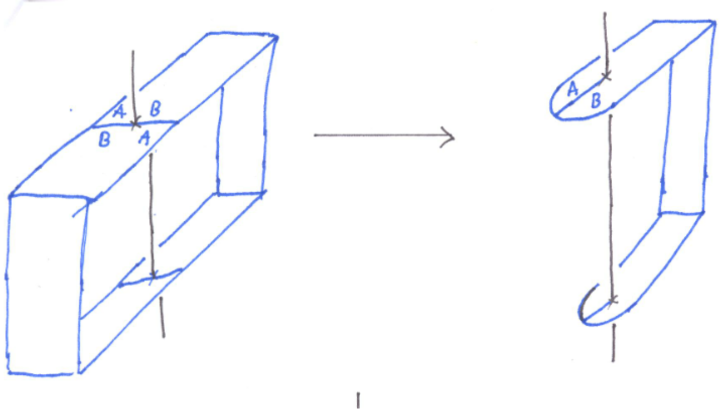


FIGURE 73. The covering map of D_1^3 as a thickened version of the map in Figure 72

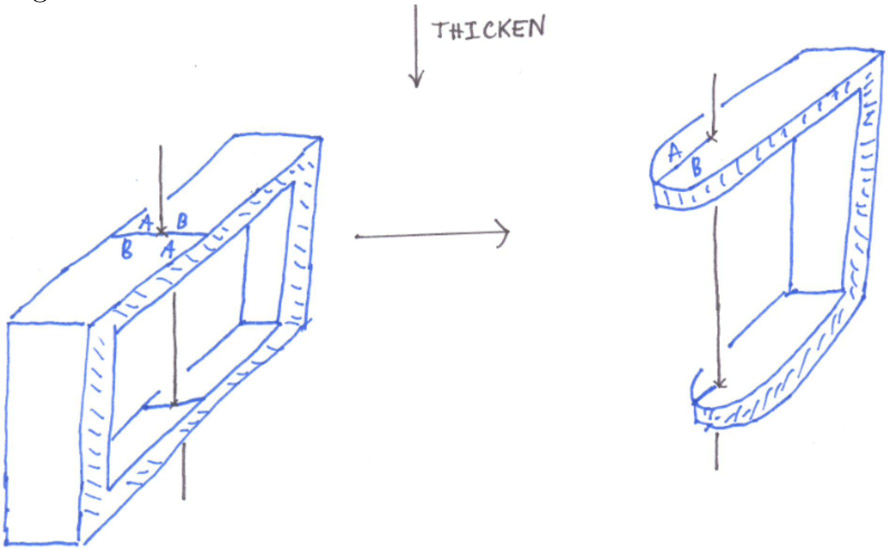
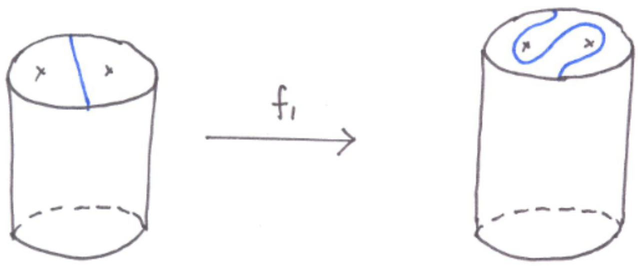


FIGURE 74. The map f_1



the twist (see Figure 75). Similarly, we can extend the map g from Example 8 to a map g_1 of the Cartesian product of the annulus with $[0, 1]$, i.e. a solid torus (see Figure 76).

Now in the base $\mathbb{S}^3$ we remove D_1^3 and glue it back in via the homeomorphism f_1, which in the covering $\mathbb{S}^3$ corresponds to removing D_2^3 and T_2^3 and gluing them back in. Note that D_2^3 is glued back in via f_1 and T_2^3 is glued back in via g_1. Figure 77 shows the effect of the gluing in the base $\mathbb{S}^3$, the shaded region represents the part where f_1 is not the

Figure 75. The positive and negative crossings

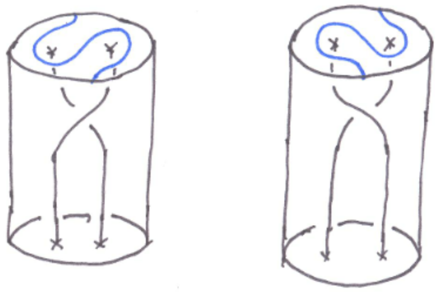
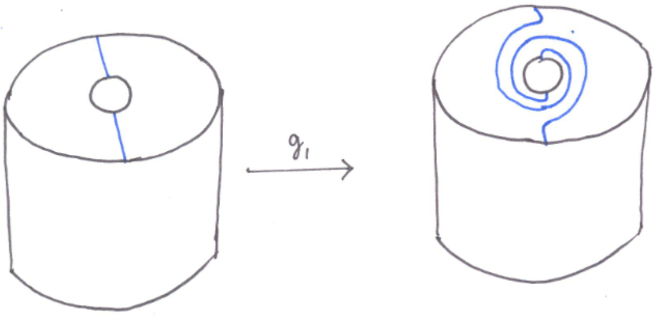
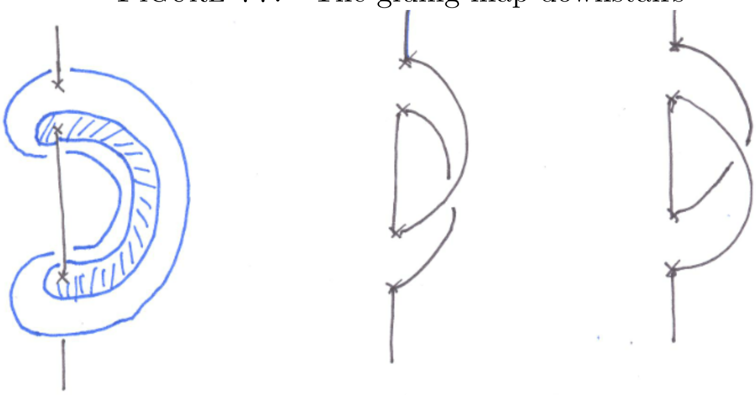


Figure 76. The map g_1



identity. Notice how the branch set is changing. After the gluing, the base manifold is

Figure 77. The gluing map downstairs



still $\mathbb{S}^3$, since removing and gluing a 3-ball does not change the manifold (by Proposition 2). In the covering $\mathbb{S}^3$, removing and gluing D_2^3 has no effect. However, removing and gluing T_2^3 will change the manifold. Figure 78 shows the map g_1 that sends a meridian of T_2^3 to a meridian $\pm$ a longitude of T_2^3 and so the effect of gluing corresponds to surgery along the unknot with surgery coefficient ± 1.

Step 3: We're now ready to prove the theorem. From Theorem 3 we know that M can be obtained from $\mathbb{S}^3$ by surgery along some link L with surgery coefficients ± 1. Let $\mathbb{S}^3 = H_1 \cup_h H_2$ be a Heegaard splitting of genus g for $\mathbb{S}^3$. Assume that H_2 is standardly embedded in $\mathbb{S}^3$ and the link L is contained in the region given as in Figure 35. Suppose L is given as in Figure 79. From **Step 2**, we know that removing and gluing 3-balls in the base $\mathbb{S}^3$ corresponds to surgery in the covering $\mathbb{S}^3$. To finish the proof, we just need

FIGURE 78. The map g_1 corresponds to surgery with coefficient ± 1

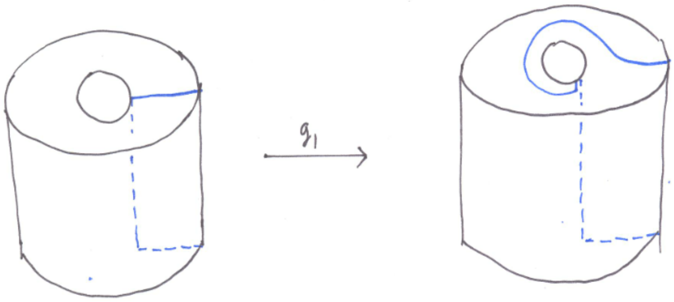
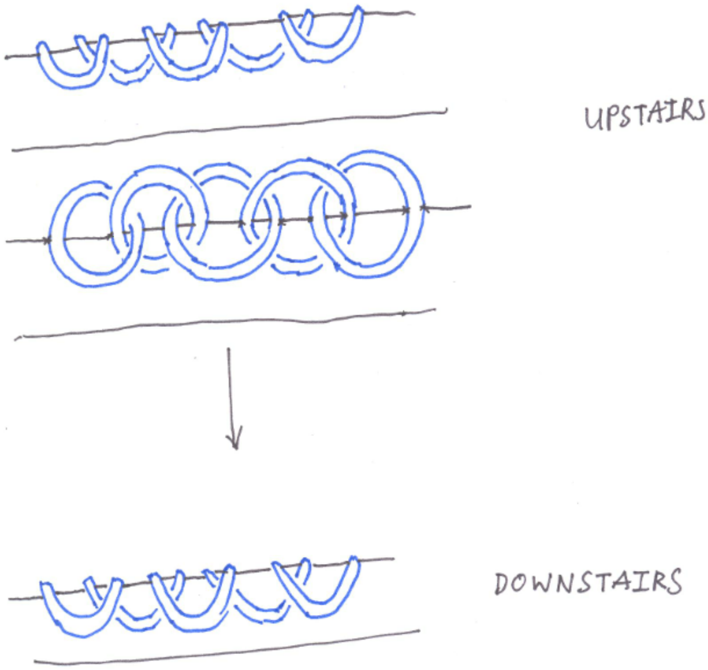


FIGURE 79. The surgery link L

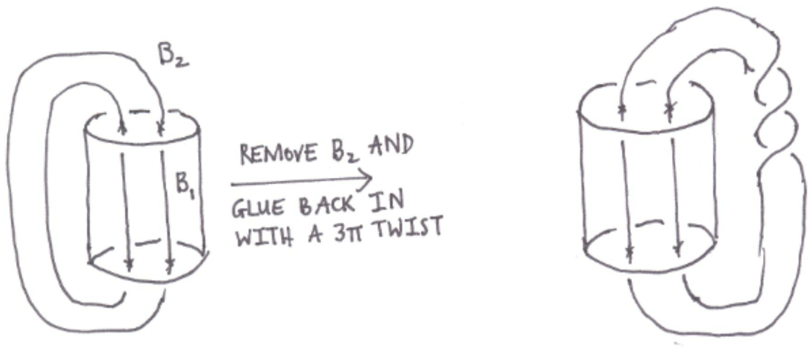
to position the 3-balls in the base $\mathbb{S}^3$ in such a way that their preimages in the covering $\mathbb{S}^3$ gives us L. See Figure 80 for an illustration for this particular link L. This gives us a 3-fold branched covering $p\colon M \to \mathbb{S}^3$.

FIGURE 80. The 3-fold branched covering



Finally, we need to make the branch set downstairs connected, i.e. a knot. Consider the 3π twist map f of the 2-disk as defined in Example 9. Since f is identical on the boundary, we can extend f to the map f_1 of the boundary of the Cartesian product of the 2-disk and $[0,1]$. The map f_1 lifts to the map g_1, which is the extension of the map g from Example 9. Now in the base $\mathbb{S}^3$ we remove a 3-ball and glue it back in via the homeomorphism f_1 of the boundary 2-sphere. In the covering manifold M, this corresponds to removing a 3-ball and gluing it back in via the homeomorphism g_1 of the boundary 2-sphere. This operation does not change the homeomorphism types of the manifolds involved. However, after the operation, the branch set downstairs will obtain a 3π twist (see Figure 81), which renders it connected (see Example 9 for a reason why we cannot just apply a π twist). $\square$

FIGURE 81. Making the branched set connected

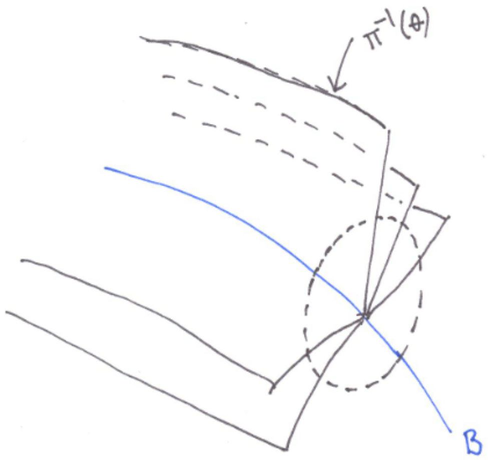


6. Open Book Decomposition

Definition 7. Let M be a closed oriented 3-manifold. An *open book decomposition* of M is a pair (B,π) where

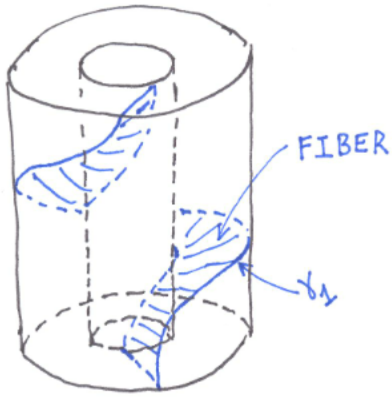
(1) B is an oriented link in M called the *binding* of the open book and
(2) $\pi\colon M-B\to\mathbb{S}^1$ is a fibration such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta\subset M$ and $\partial\Sigma_\theta=B$ for all $\theta\in\mathbb{S}^1$. The surface $\Sigma=\Sigma_\theta$, for any θ, is called the *page* of the open book. We also require the fibration to be well-behaved near B, i.e. B has a tubular neighborhood $N\times\mathbb{D}^2$ so that π restricted to $N\times(\mathbb{D}^2-0)$ is the map $(x,y)\mapsto y/|y|$, or $(x,re^{i\varphi})\mapsto\varphi$ in polar coordinates, (see Figure 82).

FIGURE 82. The local picture of an open book decomposition



Example 13. Let $M = \mathbb{S}^1 \times \mathbb{S}^2$ and $B = \mathbb{S}^1 \times \{N, S\}$, where N, S are the north and south poles of $\mathbb{S}^2$. We can construct a family of open book decompositions of (M, B) as follows. Notice that $\mathbb{S}^2 - \{N, S\}$ is homeomorphic to $\mathbb{S}^1 \times (0, 1)$. Thus $M - B$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times (0, 1)$, which is a thickened torus. Now let γ_n be a curve that runs once around the longitude and n times around the meridian of the torus. We can fiber $M - B$ by annuli parallel to $\gamma_n \times (0, 1)$ (see Figure 83).

FIGURE 83. Fibering by annuli



Example 14. The binding of an open book decomposition of $\mathbb{S}^3$ is called a *fibered link*. The simplest open book decomposition of $\mathbb{S}^3$ is when B is the unknot U. Think of $\mathbb{S}^3 - U$ as $\mathbb{R}^3 - z$-axis. In cylindrical coordinates, the fibration $\pi_U \colon \mathbb{S}^3 - U \to \mathbb{S}^1$ is given by

$$\pi_U(\rho, \theta, z) = \theta,$$

where $x = \rho \cos \theta$, $y = \rho \sin \theta$.

Theorem 5. *Every connected closed oriented 3-manifold has an open book decomposition.*

Proof. We sketch two proofs of this theorem. The first proof uses branched covering and the second proof uses surgery.

Proof 1: For this proof we need the fact that every connected closed oriented 3-manifold M is a branched cover of $\mathbb{S}^3$ along some link L_M (see Theorem 4). We also need the following:

Every link L in $\mathbb{S}^3$ can be braided about the unknot, i.e. we can isotope L so that L is contained in the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ standardly embedded in $\mathbb{S}^3$ and L is traverse to $\{p\} \times \mathbb{D}^2$ for all $p \in \mathbb{S}^1$.

Now let $p \colon M \to \mathbb{S}^3$ be a branched covering map along $L_M \in \mathbb{S}^3$. We can braid L_M about the unknot U. Set $B = p^{-1}(U) \subset M$. Then M has an open book decomposition (B, π), where $\pi \colon M - B \to \mathbb{S}^1$ is given by $\pi = \pi_U P$, where $\pi_U : \mathbb{S}^3 - U \to \mathbb{S}^1$ is the fibering of $\mathbb{S}^3 - U$ as defined in Example 14. (Note that the reason we want to braid L_M about the unknot is to make sure that each fiber of π_U intersects L_M transversely in a finite number of points and so $\pi^{-1}(\theta)$ is a branched cover of $\pi_U^{-1}(\theta)$.)

Proof 2: By Theorem 3, there are disjoint solid tori $V_1, ..., V_r$ in $\mathbb{S}^3$ and $V_1', ..., V_r'$ in M and a homeomorphism

$$h \colon \mathbb{S}^3 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to M - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r')$$

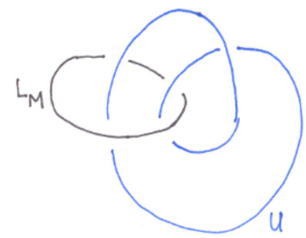
which sends the meridian of V_i to meridian $\pm$ longitude of V_i'. We may assume that each V_i wraps once around some fixed unknot U in $\mathbb{S}^3$ (see Figure 35). Let C_i denote the core of V_i'. We'll show that $h(U) \cup C_1 \cup \cdots \cup C_r$ is the binding of an open book decomposition of M. Consider the standard fibration $\pi_U : \mathbb{S}^3 - U \to \mathbb{S}^1$. Then

$$\pi_U h^{-1} : M - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r' \cup h(U)) \to \mathbb{S}^1$$

is a fibration. Note that since the fibers of $\pi_U : \mathbb{S}^3 \to \mathbb{S}^1$ intersect each V_i in a meridional disk (because of the way we arrange the link), the fibers of $\pi_u h^{-1}$ intersect each $\partial V_i'$ in a meridian $\pm$ longitude. We can therefore extend $\pi_u h^{-1}$ to a fibration of $M - (C_1 \cup \cdots \cup C_r \cup h(U))$ by annuli as in Example 13. □

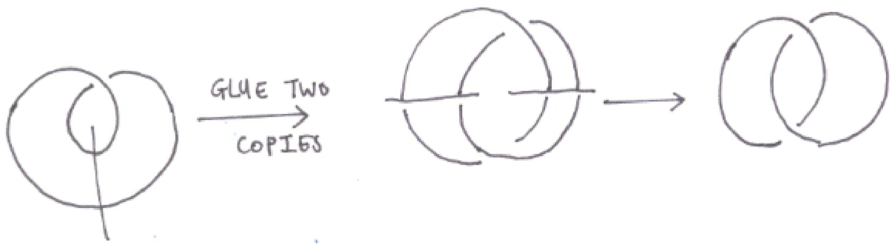
Example 15. The idea used in Proof 1 of the above theorem allows us to construct some fibered links. Let L_M and U be positioned as in Figure 84 and consider the cyclic branched cover of $\mathbb{S}^3$ along L_M. Since L_M is the unknot, the covering manifold is still

FIGURE 84. The link L_M and U



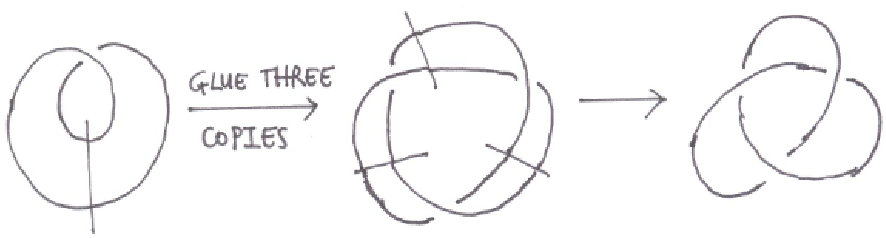
$\mathbb{S}^3$. The preimage of U in the branched cover gives us the binding of an open book decomposition of $\mathbb{S}^3$ upstairs. For two-fold cyclic branched covering, we obtain the Hopf link (Figure 85). For three-fold cyclic branched covering, we obtain the trefoil (Figure

FIGURE 85. The Hopf link is fibered



86). For a more interesting example, consider L_M and U as in Figure 87. The preimage

FIGURE 86. The trefoil is fibered



of U in the two-fold cyclic branched cover of $\mathbb{S}^3$ gives us the figure-eight-knot (Figure 88). Thus the figure-eight-knot is fibered.

FIGURE 87. Another position of U

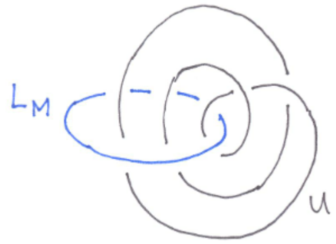
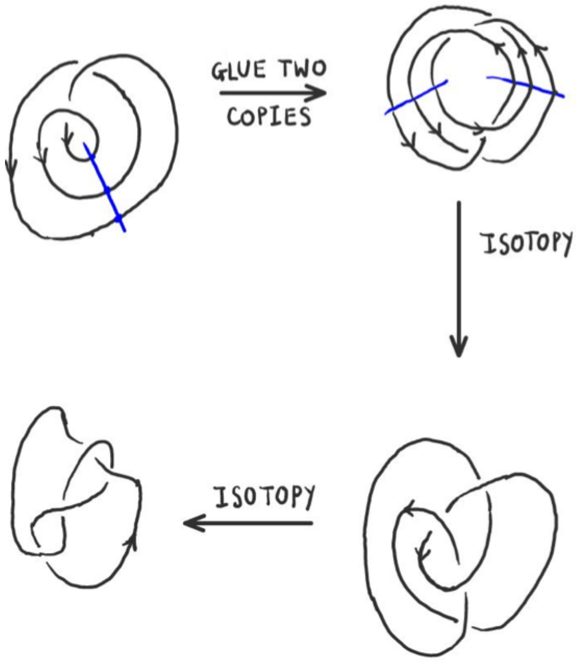


FIGURE 88. The figure-eight knot is fibered



7. From Open Book to Contact Structure

In this section we consider smooth oriented closed (compact and without boundary) 3-manifolds. We use the "outward normal first" convention to induce the orientation on the boundary of a manifold, i.e. if $(v_1, ..., v_{n-1})$ is an oriented basis for ∂N if $(\nu, v_1, ..., v_{n-1})$ is an oriented basis for N, where ν is a normal vector to ∂N that points outwards.

Definition 8. A *contact structure* ξ on M is an plane field $\xi \subset TM$ for which there is a 1-form α such that $\xi = \ker \alpha$ and $\alpha \wedge d\alpha > 0$. Such a 1-form α is called a *contact form* and (M, ξ) is called a *contact manifold.*

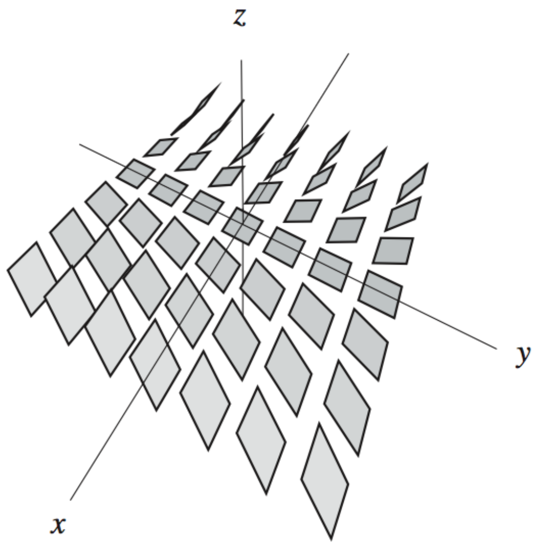
Thus we can only define a contact structure on M if M is orientable. The requirement $\alpha \wedge d\alpha > 0$ also specifies an orientation on M.

Example 16. The *standard contact structure* on $\mathbb{R}^3$ is given by $\ker \alpha$, where $\alpha = dz + xdy$ (see Figure 89). We have

$$\alpha \wedge d\alpha = (dz + xdy) \wedge dx \wedge dy = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz > 0,$$

where we choose the standard orientation of $\mathbb{R}^3$. At a point (x, y, z), the contact plane ξ is spanned by $\{\partial_x, x\partial_z - \partial_y\}$. So at any point in the yz-plane (where $x = 0$) ξ_1 is horizontal. If we start at $(0, 0, 0)$ we have a horizontal plane and as we move out along the x-axis the plane will twist in a clockwise direction. The twist will be by $\pi/2$ when x goes to ∞. There is a similar behavior on all rays perpendicular to the yz-plane.

FIGURE 89. The standard contact structure on $\mathbb{R}^3$



Before we proceed, we describe another way to present an open book decomposition of a 3-manifold, known as *abstract open book*.

Definition 9. An abstract open book is a pair (Σ, ϕ), where
(1) Σ is an oriented compact surface with boundary and
(2) $\phi\colon \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial\Sigma$. The map ϕ is called the *monodromy*.

Given an abstract open book (Σ, ϕ), where $\partial\Sigma = \cup_i \mathbb{S}^1$, we can obtain a 3-manifold $M(\phi)$ as follows. First, the *mapping torus* $\Sigma(\phi)$ is the quotient space obtained from $\Sigma \times [0, 2\pi]$ by identifying $(x, 2\pi)$ with $(\phi(x), 0)$. Note that $\Sigma(\phi)$ is a 3-manifold with boundary $\cup_i \mathbb{S}^1 \times \mathbb{S}^1$ (here $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$). To obtain $M(\phi)$, we glue a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ to each boundary component using the identity map (see Figure 90). Let B denote $\partial\Sigma \times \{0\}$, we define a fibration $p\colon M(\phi) - B \to \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ by

$$p([x, \varphi]) = [\varphi] \text{ for } [x, \varphi] \in \Sigma(\phi)$$

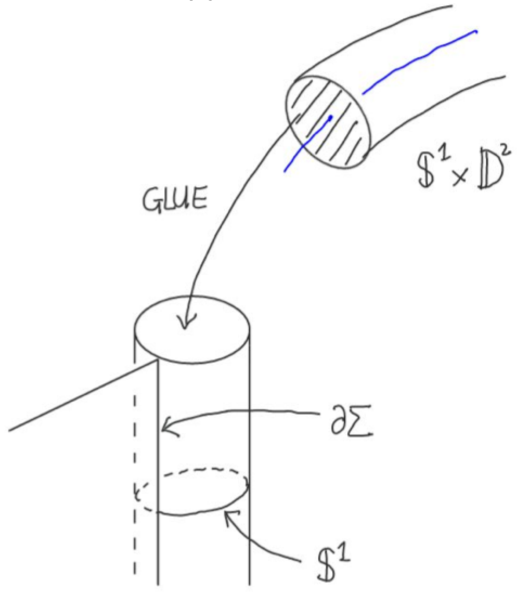
and

$$p(\theta, re^{i\varphi}) = [\varphi] \text{ for } (\theta, re^{i\varphi}) \in \partial\Sigma \times \mathbb{D}^2.$$

Conversely, the abstract open book can be recovered from an open book decomposition $p\colon M - B \to \mathbb{S}^1$ as follows. Define Σ as the intersection of the page $p^{-1}(0)$ with the complement of an open tubular neighborhood $B \times \mathrm{Int}(\mathbb{D}^2_{1/2})$ (here $\mathbb{D}^2_r$ means a 2-disk of radius r). To find the monodromy ϕ, choose a Riemannian metric on M such that the vector field ∂_φ on $B \times (\mathbb{D}^2 - \{0\})$ is orthogonal to the pages. Extend this to a vector field on $N - B$ orthogonal to the pages that projects under the differential dp to the vector field ∂_φ on $\mathbb{S}^1$. Consequently we obtain a vector field on N vanishing along B. The flow of this vector field after time 2π gives us a diffeomorphism of Σ and that is our required monodromy ϕ.

Let us look at a few examples on how to obtain $M(\phi)$ from an abstract open book (Σ, ϕ).

Figure 90. Fill in the core



Example 17. Let $\Sigma = \mathbb{D}^2$ and ϕ is the identity map. The mapping cylinder $\Sigma(\phi)$ in this case is just a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$. Its boundary is $\mathbb{S}^1 \times \mathbb{S}^1$. To obtain $M(\phi)$, we glue in $\mathbb{S}^1 \times \mathbb{D}^2$ via the identity map. Notice that the meridian of the second torus is a longitude of the first torus. The 3-manifold thus obtained is $\mathbb{S}^3$.

Example 18. Let Σ be an annulus, and ϕ is the identity map. Here Σ has two boundary components. Let a be the outer circle and b be the inner circle. Then $\Sigma(\phi)$ is a solid torus with an inner solid torus removed. To obtain $M(\phi)$, we need to glue two solid tori to $a \times \mathbb{S}^1$ and $b \times \mathbb{S}^1$. After we glue a solid torus to $a \times \mathbb{S}^1$, we obtain $\mathbb{S}^3$ with a solid torus removed, hence a solid torus. Note that b is a longitude of this solid torus. Now we glue another solid torus to $b \times \mathbb{S}^1$. Note that the meridian of the second torus is glued to the meridian of the first solid torus. Thus we obtain $\mathbb{S}^1 \times \mathbb{S}^2$. The binding in this case is the trivial link with two components.

Now suppose ϕ is a Dehn twist. Let p be a point on the inner circle a and q be a point on the outer circle b. Form the mapping torus $\Sigma(\phi)$. The key thing is to see what happens to $\{p\} \times \mathbb{S}^1$ and $\{q\} \times \mathbb{S}^1$. After the twist, $\{q\} \times \mathbb{S}^1$ is still a longitude. The result of gluing a solid torus to $\{q\} \times \mathbb{S}^1$ gives $\mathbb{S}^3$ with a solid torus removed, which is a solid torus, say T. Note that due to Dehn twist, $\{p\} \times \mathbb{S}^1$ is meridian $\pm$ longitude of T. Thus the result of gluing a solid torus to T via a homeomorphism that sends a meridian to $\{p\} \times \mathbb{S}^1$ is $\mathbb{S}^3$ (this is just surgery with coefficient ± 1). In this case the binding is the Hopf link.

Definition 10. Let M be an oriented closed 3-manifold with an open book decomposition (B, p), where we require that the binding B and the pages $p^{-1}(\varphi)$ are oriented accordingly, i.e. ∂_φ together with the orientation of the page gives the orientation of M. A contact structure $\xi = \ker \alpha$ is said to be *supported* by the open book decomposition (B, p) of M if

(i) the 2-form $d\alpha$ is an area form on each page
(ii) $\alpha > 0$ on B (α specifies an orientation on B).

Remark 4. Two contact structures ξ_0 and ξ_1 are called *isotopic* if there is a 1-parameter family of contact structures connecting them. A more intuitive description of contact structures supported by open book is given as follows: the contact manifold (M, ξ) is supported by the open book (B, π) if ξ can be isotoped to be arbitrarily close, on compact subsets of the pages, to the tangent planes to the pages of the open book in such a way that after some point in the isotopy the contact planes are transverse to B and transverse to the pages of the open book in a fixed neighborhood of B. For the equivalence of this description and the above description, see [3, Lemma 3.5].

Example 19. Regard $\mathbb{S}^3$ as the subset of $\mathbb{C}^2$ given by

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}.$$

In polar coordinates $(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2})$, the *standard contact form* on $\mathbb{S}^3$ is given by

$$\alpha = r_1^2 d\varphi_1 + r_2^2 d\varphi_2.$$

Set

$$B = \{(z_1, z_2) \in \mathbb{S}^3 \colon z_1 = 0\}.$$

So B is the unknot and consider the standard fibration of $\mathbb{S}^3$

$$p \colon \mathbb{S}^3 - B \to \mathbb{S}^1 \text{ given by } (z_1, z_2) \mapsto \frac{z_1}{|z_1|},$$

or in polar coordinates

$$p(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \varphi_1.$$

Note that the page is an open 2-disk:

$$p^{-1}(\varphi_1) = \{(r_2, \varphi_2) \colon r_2 < 1\}.$$

The 1-form α restricts to $d\varphi_2$ along B and on a page, where φ_1 is constant, we have

$$d\alpha = 2r_2 dr_2 \wedge d\varphi_2,$$

which is the volume form on the disk. So the contact structure on $\mathbb{S}^3$ is supported by the open book decomposition.

Theorem 6 (Thurston-Winkelnkemper)**.** *Every open book decomposition (Σ, ϕ) supports a contact structure ξ_ϕ on $M(\phi)$.*

Proof. Choose an orientation for Σ, and give $\partial\Sigma$ the induced boundary orientation (outward normal first). We assume that $\partial\Sigma = \mathbb{S}^1$. Let $\theta \in \mathbb{S}^1$ be the coordinate along $\partial\Sigma$, and s a collar parameter for $\partial\Sigma$ in Σ, with $\partial\Sigma = \{s = 0\}$ and $s < 0$ in the interior of Σ. We also assume that s has been chosen so that $\phi \colon \Sigma \to \Sigma$ is the identity on $[-2, 0] \times \partial\Sigma$.

The idea of the proof is quite straightforward. We first construct a 1-form α on $\Sigma(\phi)$, and then extend the 1-form α to the core solid torus. We need to make sure that α is smooth and is a contact form. The details are a bit involved, so we divide the proof into two steps.

Step 1: We construct a 1-form α on $\Sigma(\phi)$. For that we first show that the set of 1-forms β on Σ satisfying

(i) $\beta = e^s d\theta$ on $[-3/2, 0] \times \partial\Sigma$,
(ii) $d\beta$ is an area form on Σ of total area 2π.

is non-empty and convex.

To see why, start with an arbitrary 1-form β_0 on Σ with $\beta_0 = e^s d\theta$ on the collar $[-2, 0] \times \partial\Sigma$. Then by Stokes's theorem

$$\int_\Sigma d\beta_0 = \int_{\partial\Sigma} \beta_0 = \int_{\partial\Sigma} d\theta = 2\pi.$$

(Note that $s = 0$ on $\partial\Sigma$.) Let ω be any area form on Σ of total area 2π and with $\omega = e^s ds \wedge d\theta$ on the collar. Then

$$\int_\Sigma \omega - d\beta_0 = \int_\Sigma \omega - \int_\Sigma d\beta = 0 \text{ and}$$

$$\omega - d\beta_0 = e^s ds \wedge d\theta - e^s ds \wedge d\theta = 0 \text{ on } [-2, 0] \times \partial\Sigma.$$

Now recall that the compactly supported second de Rham cohomology of the open surface

$$\Sigma - \left(\left[-\frac{3}{2}, 0\right] \times \partial\Sigma\right)$$

is isomorphic to $\mathbb{R}$, and the isomorphism is given by

$$\omega \mapsto \int_\Sigma \omega$$

(See [8], Theorem 15.20). Since $\omega - d\beta_0 \equiv 0$ on $[-2, 0] \times \partial\Sigma$, it belongs to the above cohomology group. Moreover, by the isomorphism, $\omega - d\beta_0 = 0$ in the cohomology group, i.e. there exists a 1-form β_1 vanishing on $[-3/2, 0] \times \partial\Sigma$ such that $\omega - d\beta_0 = d\beta_1$. Let $\beta \colon = \beta_0 + \beta_1$. Then clearly $\beta = e^s d\theta$ on $[-3/2, 0] \times \partial\Sigma$ and

$$d\beta = d\beta_0 + d\beta_1 = d\beta_0 + \omega - d\beta = \omega,$$

which is an area form on Σ of total area 2π by construction. It follows that the set of 1-forms β satisfying conditions (i) and (ii) is non-empty. Now for any other 1-form β' satisfying conditions (i) and (ii), consider the 1-form $\beta_t = (1-t)\beta + t\beta'$ for some $t \in [0, 1]$. Then clearly $\beta_t = e^s d\theta$ on $[-3/2, 0] \times \partial\Sigma$. Moreover,

$$d\beta_t = (1-t)d\beta + td\beta'$$

is also an area form on Σ of total area 2π (since the functions of t are non-negative and do not vanish at the same time). Therefore the set of 1-forms β is also convex.

Let β be any 1-form as just described. Let $\mu \colon [0, 2\pi] \to [0, 1]$ be a smooth function that is identically 1 near $\varphi = 0$ and identically 0 near $\varphi = 2\pi$. Consider the 1-form

$$\mu(\varphi)\beta + (1 - \mu(\varphi))\phi^*\beta$$

on $\Sigma \times [0, 2\pi]$. Notice that at $(x, 2\pi)$ the 1-form becomes $\phi^*\beta_x = \beta_{\phi(x)}$ and at $(\phi(x), 0)$ the 1-form becomes $\beta_{\phi(x)}$. Therefore it induces a 1-form on $\Sigma(\phi)$ (recall that we identify $(x, 2\pi)$ with $(\phi(x), 0)$). We denote this 1-form on $\Sigma(\phi)$ by $\widetilde{\beta}$. Since ϕ is the identity on a collar of $\partial\Sigma$, on $[-3/2, 0] \times \partial\Sigma$ we have $\phi^*\beta = \beta = e^s d\theta$. Moreover, $d\phi^*\beta$ is still an area form on Σ because ϕ is an orientation-preserving diffeomorphism. The total area is given by

$$\int_\Sigma d\phi^*\beta = \int_{\partial\Sigma} \phi^*\beta = \int_{\partial\Sigma} \beta = 2\pi.$$

By convexity we conclude that for each fixed φ, $\widetilde{\beta}$ is a 1-form on the page over φ that satisfies conditions (i) and (ii) specified above. In particular we have $\widetilde{\beta} = e^s d\theta$ on the collar

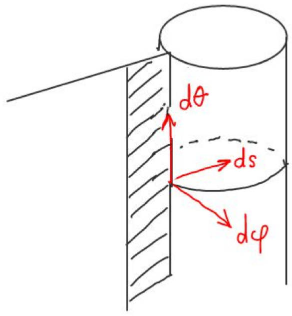
$$\left[-\frac{3}{2}, 0\right] \times \partial\Sigma(\phi) = \left[-\frac{3}{2}, 0\right] \times \partial\Sigma \times \mathbb{S}^1.$$

Now put $\alpha = \widetilde{\beta} + Cd\varphi$ for some positive constant C. Then

$$\alpha \wedge d\alpha = (\widetilde{\beta} + Cd\varphi) \wedge d\widetilde{\beta} = \widetilde{\beta} \wedge d\widetilde{\beta} + Cd\varphi \wedge d\widetilde{\beta}.$$

Note that $d\varphi \wedge d\widetilde{\beta}$ is the volume form on $\Sigma(\phi)$. Since Σ is compact, we can choose C big enough so that $\alpha \wedge d\alpha > 0$. We've obtained a contact form on $\Sigma(\phi)$.

FIGURE 91. The standard coordinates on an abstract open book



Step 2: We extend this α to a contact form on $M(\phi)$, i.e. we need to extend α to the core solid torus $\partial\Sigma \times \mathbb{D}^2$. To ensure smoothness of α, we create some overlap between the core solid torus and the pages. Specifically, we represent $M(\phi)$ as

$$M(\phi) \cong (\Sigma(\phi) \sqcup (\partial\Sigma \times \mathbb{D}_2^2))/\sim,$$

where in the equivalence relation we identify

$$(s, \theta, \varphi) \in [-1, 0] \times \partial\Sigma \times \mathbb{S}^1$$

with

$$(\theta, r = 1 - s, \varphi) \in \partial\Sigma \times [1, 2] \times \mathbb{S}^1 \subset \partial\Sigma \times \mathbb{D}_2^2.$$

Pictorially, we glue the $[-1, 0]$-part of the page with the $[1, 2]$-part of the core (in other words we push the pages slightly into the core). Now inside the core $\partial\Sigma \times \mathbb{D}_2^2$ we put

$$\alpha = h_1(r)d\theta + h_2(r)d\varphi,$$

where we require that

(1) $h_1(r) = 2 - r^2$ and $h_2(r) = r^2$ near $r = 0$,
(2) $h_1(r) = e^{1-r}$ and $h_2(r) = C$ for $r \in [1, 2]$.

Notice that on the overlap of the pages and the core α is given by

$$\alpha = e^{1-r}d\theta + Cd\varphi = e^s d\theta + Cd\varphi,$$

which agrees with the definition of α defined in Step 1. To check that α is a contact form, note that

$$\begin{aligned}\alpha \wedge d\alpha &= (h_1(r)d\theta + h_2(r)d\varphi) \wedge (h_1'(r)dr \wedge d\theta + h_2'(r)dr \wedge d\varphi)\\ &= (h_2(r)h_1'(r) - h_1(r)h_2'(r))d\varphi \wedge dr \wedge d\theta.\end{aligned}$$

So we require $h_1(r)$, $h_2(r)$ to satisfy $|h_2(r)h_1'(r) - h_1(r)h_2'(r)| > 0$, i.e.

(3) $(h_1(r), h_2(r))$ is not parallel with $(h_1'(r), h_2'(r))$.

It's not hard to see that there exist $h_1(r)$, $h_2(r)$ that satisfy conditions (1), (2), (3).

Now we verify that the contact structure is indeed supported by the open book. Near $r = 0$ the 1-form α is given by

$$\alpha = (2 - r^2)d\theta + r^2 d\varphi.$$

So α restricted to the binding (i.e. $r = 0$) is $2d\theta$, which verifies condition (ii) in Definition 10. For condition (i) in Definition 10, we restrict $d\alpha$ to a page $\varphi =$ constant. Outside the core solid torus, $d\alpha = d\widetilde{\beta}$, which is an area form on the page by definition of $\widetilde{\beta}$. Inside the core solid torus, we have

$$d\alpha = h_1'(r)dr \wedge d\theta = -h_1'(r)ds \wedge d\theta.$$

Since $h_1'(r) < 0$, $d\alpha$ is an area form (here we choose $ds \wedge d\theta$ as the standard orientation of the page) and we conclude that α is our required contact form. $\square$

8. Conclusion

The connection between open book decomposition and contact geometry is provided by the following fundamental theorem of Giroux.

Theorem 7 (Giroux)**.** *Let M be a closed oriented 3-manifold. Then there is a one to one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilization.*

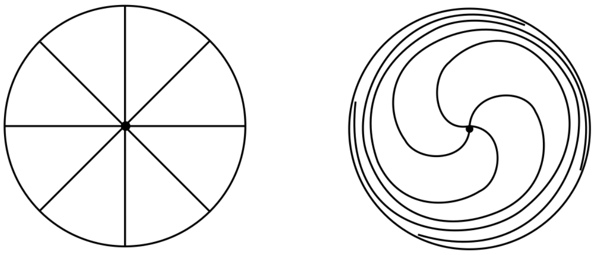
Here a positive stabilization of an abstract open book (Σ, ϕ) is the open book (Σ', ϕ'), where $\Sigma' = \Sigma \cup$ 1-handle and $\phi' = \phi \circ \tau_C$, where C is a curve that intersects the co-core of the 1-handle exactly one time. The easy direction of the theorem is given in parts by Theorem 6. For the other more difficult direction, the readers are referred to [3] for more details.

Let F be an embedded oriented surface in in a contact manifold (M, ξ). At each point $x \in F$ consider

$$l_x = \xi_x \cap T_x F.$$

For most x, the subspace l_x will be a line in $T_x F$, but at some points, which we call *singular points*, $l_x = T_x F$, i.e. the tangent space at x coincides with the contact plane. The *characteristic foliation* of F is the foliation $\mathcal{F}$ on the complement of these singularities obtained by integrating along l_x. We denote the characteristic foliation of F by $\mathcal{F}_\xi(F)$. There is a fundamental dichotomy in 3-dimensional contact geometry. A contact structure ξ on M is called *overtwisted* if there is an embedded disk whose characteristic foliation is homeomorphic to either one shown in Figure 92 (the one on the right is obtained by pushing the interior of the disk up slightly). Otherwise it is called *tight*.

Figure 92. The overtwisted disk



Recently, there has been a new technique that uses open book decomposition to study the geometry and topology of 3-manifolds known as *open book foliation*. Open book foliation has its origin in the theory of braid foliation, as developed by Birman and Menasco. Consider the standard open book decomposition of $\mathbb{S}^3$, with binding $Oz \cup \{\infty\}$, where Oz denotes the z-axis. A link L in the complement of the binding can be braided about Oz. Consider an incompressible surface F bounded by L. The intersection of F and the pages of the open book induces a singular foliation $\mathcal{F}$ on F. We require the foliation $\mathcal{F}$ to satisfy certain properties, which can be achieved by putting F in a "nice" position. The foliation $\mathcal{F}$ is called the *braid foliation*. Roughly speaking, open book foliation is a generalization of braid foliation to general 3-manifolds. So an open book decomposition of a 3-manifold induces a foliation on a surface F embedded in the 3-manifold. Open book foliation can be used to prove certain classical results in 3 dimensional contact geometry, as well as provide new insights into the geometry and topology of 3-manifolds. For a relationship between characteristic foliation and open book foliation, as well as many exciting developments in this direction, the readers can consult the recent papers [5] and [6].

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