The .nb (source) file is at http://www.math.toronto.edu/vohuan/, based on the original program at http://-drorbn.net/AcademicPensieve/2015-07/PolyPoly/.

Alexander Invariant of Tangles

1. Γ-Calculus

The following code expresses an element of Γ calculus in a nice format.

In[1]:=

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
 Collect[λ, x_, Collect[#,y_, Factor]&]];
Format[Γ[ω_,λ_]] := Module[{S, M},
 S = Union@Cases[Γ[ω,λ], (x|y)_a_ :→ a, ∞];
 M = Outer[Factor[∂_{x#1 y#2}λ]&, S, S];
 M = Prepend[M, y_#& /@ S] // Transpose;
 M = Prepend[M, Prepend[x_#& /@ S, ω]];
 M // MatrixForm];
```

For instance, we can display an element of Γ calculus as follows.

In[3]:=

$$\Gamma\left[\omega,\{y_a,y_b\}.\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}.\{x_a,x_b\}\right]$$

Out[3]=

$$\begin{pmatrix} \omega & x_a & x_b \\ y_a & g_{11} & g_{12} \\ y_b & g_{21} & g_{22}\end{pmatrix}$$

Next we program the stitching operation

In[4]:=

$$\Gamma \;/:\; \Gamma[\omega 1\_,\ \lambda 1\_]\Gamma[\omega 2\_,\ \lambda 2\_] \;:=\; \Gamma[\omega 1 * \omega 2,\ \lambda 1+\lambda 2];$$

$$m_{a\_,e\_\to c\_}[\Gamma[\omega\_,\ \lambda\_]] \;:=\; \text{Module}\Big[\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu\},$$

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{y_a,x_a}\lambda & \partial_{y_a,x_e}\lambda & \partial_{y_a}\lambda \\ \partial_{y_e,x_a}\lambda & \partial_{y_e,x_e}\lambda & \partial_{y_e}\lambda \\ \partial_{x_a}\lambda & \partial_{x_e}\lambda & \lambda \end{pmatrix} \;/.\; (y|x)_{a|e}\to 0;$$

$$\Gamma\left[(\mu=1-\gamma)\,\omega,\ \{y_c,1\}.\begin{pmatrix} \beta+\alpha\ \delta/\mu & \theta+\alpha\ \epsilon/\mu \\ \psi+\delta\ \phi/\mu & \Xi+\phi\ \epsilon/\mu \end{pmatrix}.\{x_c,1\}\right]$$

$$/.\ \{t_a\to t_c,\ t_e\to t_c\}\ //\ \Gamma\text{Collect}\Big];$$

$$R^{+}_{a\_,e\_} \;:=\; \Gamma\left[1,\ \{y_a,y_e\}.\begin{pmatrix} 1 & 1-t_a \\ 0 & t_a \end{pmatrix}.\{x_a,x_e\}\right];$$

$$R^{-}_{a\_,e\_} \;:=\; R^{+}_{a,e}\ /.\ t_a\to 1/t_a;$$

Checking meta-associativity

In[8]:=

$$\zeta = \Gamma\left[\omega,\ \{y_1, y_2, y_3, y_S\} . \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix} . \{x_1, x_2, x_3, x_S\}\right]$$

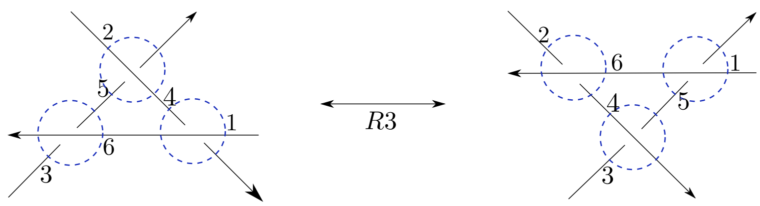
$$(\zeta // m_{1,2\to1} // m_{1,3\to1}) == (\zeta // m_{2,3\to2} // m_{1,2\to1})$$

Out[8]=

$$\begin{pmatrix} \omega & x_1 & x_2 & x_3 & x_S \\ y_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ y_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ y_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ y_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

Out[9]= True

Checking the RIII relation



In[10]:=

$$R^+_{1,4} R^+_{2,5} R^-_{6,3} // m_{1,6\to1} // m_{2,4\to2} // m_{3,5\to3}$$

Out[10]=

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ y_1 & 1 & 1 - t_1 & \frac{(-1+t_1)\, t_2}{t_1} \\ y_2 & 0 & t_1 & 1 - t_2 \\ y_3 & 0 & 0 & \frac{t_2}{t_1} \end{pmatrix}$$

In[11]:=

$$R^-_{1,4} R^+_{5,2} R^+_{6,3} // m_{1,5\to1} // m_{2,6\to2} // m_{3,4\to3}$$

Out[11]=

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ y_1 & 1 & 1 - t_1 & \frac{(-1+t_1)\, t_2}{t_1} \\ y_2 & 0 & t_1 & 1 - t_2 \\ y_3 & 0 & 0 & \frac{t_2}{t_1} \end{pmatrix}$$

Checking the RII relations

In[12]:=

$$R^+_{i,j} R^-_{k,l} // m_{i,k\to i} // m_{j,l\to j}$$

Out[12]=

$$\begin{pmatrix} 1 & x_i & x_j \\ y_i & 1 & 0 \\ y_j & 0 & 1 \end{pmatrix}$$

Checking the Overcrossings Commute (OC) relation

In[13]:=

$$R^+_{4,2} R^+_{1,3} // m_{1,4\to1}$$

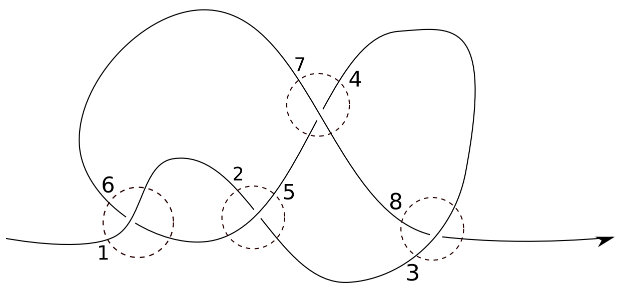
Out[13]=

$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ y_1 & 1 & 1 - t_1 & 1 - t_1 \\ y_2 & 0 & t_1 & 0 \\ y_3 & 0 & 0 & t_1 \end{pmatrix}$$

In[14]:=
$$R^{+}_{4,3}R^{+}_{1,2}//m_{1,4\to 1}$$

Out[14]=
$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ y_1 & 1 & 1-t_1 & 1-t_1 \\ y_2 & 0 & t_1 & 0 \\ y_3 & 0 & 0 & t_1 \end{pmatrix}$$

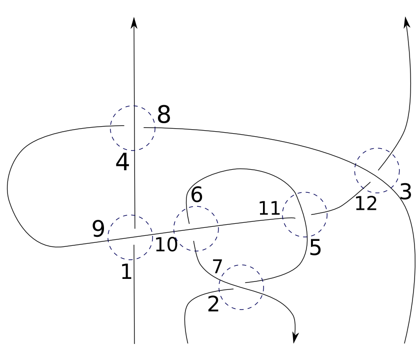
Compute the invariant of the figure-eight knot



In[15]:=
$$R^{+}_{1,6}R^{+}_{5,2}R^{-}_{3,8}R^{-}_{7,4}//m_{1,2\to 1}//m_{1,3\to 1}//m_{1,4\to 1}//m_{1,5\to 1}//m_{1,6\to 1}//m_{1,7\to 1}//m_{1,8\to 1}$$

Out[15]=
$$\begin{pmatrix} 3-\frac{1}{t_1}-t_1 & x_1 \\ y_1 & 1 \end{pmatrix}$$

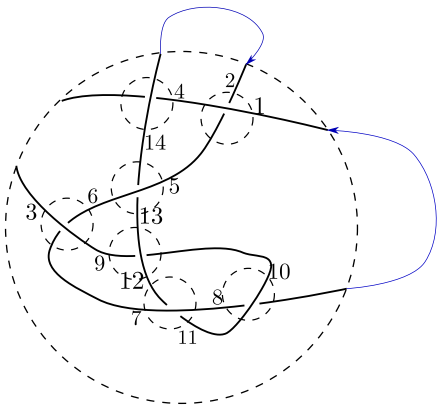
As another example, let us compute the invariant of the following tangle



In[16]:=
$$R^{+}_{7,2}\ R^{-}_{10,6}\ R^{-}_{5,11}\ R^{-}_{3,12}\ R^{+}_{4,8}\ R^{+}_{9,1}\ //\ m_{1,4\to 1}\ //\ m_{2,5\to 2}\ //\ m_{2,6\to 2}\ //\ m_{2,7\to 2}\ //\ m_{3,8\to 3}\ //\ m_{3,9\to 3}\ //\ m_{3,10\to 3}\ //\ m_{3,11\to 3}\ //\ m_{3,12\to 3}$$

Out[16]=
$$\begin{pmatrix} \left(1+\frac{-1+t_2}{t_3}\right)(-t_1(-1+t_3)+t_3) & x_1 & x_2 & x_3 \\ y_1 & -\frac{t_3}{-t_1-t_3+t_1 t_3} & \frac{(-1+t_1)(-1+t_3)t_3}{(-1+t_2+t_3)(-t_1-t_3+t_1 t_3)} & -\frac{(-1+t_1)(1-t_2-2 t_3+t_2 t_3)}{(-1+t_2+t_3)(-t_1-t_3+t_1 t_3)} \\ y_2 & 0 & \frac{t_2}{-1+t_2+t_3} & \frac{-1+t_2}{-1+t_2+t_3} \\ y_3 & \frac{t_1(-1+t_3)}{-t_1-t_3+t_1 t_3} & -\frac{t_1(-1+t_3)}{(-1+t_2+t_3)(-t_1-t_3+t_1 t_3)} & \frac{-1+t_1+t_2-t_1 t_2+2 t_3-3 t_1 t_3-t_2 t_3+t_1 t_2 t}{(-1+t_2+t_3)(-t_1-t_3+t_1 t_3)} \end{pmatrix}$$

Now let us compute the invariant of the knot 7_7 in the Knot Atlas.



First we compute the invariant of the tangle inside the circle

In[18]:=

$$\tau_1 = R^{+}_{1,2}\, R^{+}_{14,4}\, R^{-}_{5,13}\, R^{-}_{3,6}\, R^{-}_{12,9}\, R^{+}_{7,11}\, R^{+}_{10,8} \;//\; m_{1,4\to1} \;//\; m_{2,5\to2} \;//\; m_{2,6\to2} \;//\; m_{2,7\to2} \;//\; m_{2,8\to2} \;//\; m_{3,9\to3} \;//\; m_{3,10\to3} \;//\; m_{3,11\to3} \;//\; m_{3,12\to3} \;//\; m_{3,13\to3} \;//\; m_{3,14\to3}$$

Out[18]=

$$\begin{pmatrix} 1+t_2\left(-1+\frac{1}{t_3}\right) & x_1 \\ y_1 & \frac{-1+t_1+2\,t_2-2\,t_1\,t_2-t_2^2+t_1\,t_2^2+2\,t_3-2\,t_1\,t_3-4\,t_2\,t_3+4\,t_1\,t_2\,t_3+t_2^2\,t_3-2\,t_1\,t_2^2\,t_3-t_3^2+t_1\,t_3^2+t_2\,t_3^2-2\,t_1\,t_2\,t_3^2+t_1\,t_2^2\,t_3^2}{t_2\,(-t_2-t_3+t_2\,t_3)} \\ y_2 & -\frac{t_1\,(-1+t_2)^2\,(-1+t_3)^2}{t_2\,(-t_2-t_3+t_2\,t_3)} \\ y_3 & -\frac{(-1+t_3)\,(1-2\,t_2-t_3+t_2\,t_3)}{t_2\,(-t_2-t_3+t_2\,t_3)} \end{pmatrix}$$

The to obtain the invariant of the knot we perform two extra stitchings

In[19]:=

$$\tau_1 \;//\; m_{2,1\to2} \;//\; m_{3,2\to1}$$

Out[19]=

$$\begin{pmatrix} 9+\frac{1}{t_1^2}-\frac{5}{t_1}-5\,t_1+t_1^2 & x_1 \\ y_1 & 1 \end{pmatrix}$$

2. Extended Γ-Calculus

First we format extended Γ-calculus

In[20]:=

```
eΓCollect[eΓ[ω_, λ_, σ_]] := eΓ[Simplify[ω],
   Collect[λ, x_, Collect[#, y_, Factor] &], σ];
Format[eΓ[ω_, λ_, σ_]] := Module[{S, M},
   S = Union@Cases[eΓ[ω, λ, σ], (x | y)a_ :→ a, ∞];
   M = Outer[Factor[∂x#1y#2 λ] &, S, S];
   M = Prepend[M, y# & /@ S] // Transpose;
   M = Prepend[M, Prepend[x# & /@ S, ω]];
   {M // MatrixForm, σ}];
eΓ[ω1_, λ1_, σ1_] ≡ eΓ[ω2_, λ2_, σ2_] :=
  Simplify[PowerExpand[ω1 == ω2 ⋀ λ1 == λ2 ⋀ σ1 == σ2 ]];
```

For example

In[23]:=
$$e\Gamma\left[1,\ \{y_a, y_e\}.\begin{pmatrix} 1 & 1-t_a \\ 0 & t_a \end{pmatrix}.\{x_a, x_e\},\ \{s_a, s_e\}.\{v_a, v_e\}\right]$$

Out[23]=
$$\left\{\begin{pmatrix} 1 & x_a & x_e \\ y_a & 1 & 1-t_a \\ y_e & 0 & t_a \end{pmatrix},\ s_a\, v_a + s_e\, v_e\right\}$$

Extended stitching

In[24]:=
$$e\Gamma\ /:\ e\Gamma[\omega 1\_,\ \lambda 1\_,\ \sigma 1\_]\ e\Gamma[\omega 2\_,\ \lambda 2\_,\ \sigma 2\_]\ :=\ e\Gamma[\omega 1 * \omega 2,\ \lambda 1 + \lambda 2,\ \sigma 1 + \sigma 2];$$
$$em_{a\_,e\_\to c\_}[e\Gamma[\omega\_,\ \lambda\_,\ \sigma\_]]\ :=\ \text{Module}\big[\{\alpha, \beta, \gamma, \delta, \theta, \epsilon, \phi, \psi, \Xi, \mu\},$$
$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{y_a,x_a}\lambda & \partial_{y_a,x_e}\lambda & \partial_{y_a}\lambda \\ \partial_{y_e,x_a}\lambda & \partial_{y_e,x_e}\lambda & \partial_{y_e}\lambda \\ \partial_{x_a}\lambda & \partial_{x_e}\lambda & \lambda \end{pmatrix}\ /.\ (y\mid x)_{a\mid e} \to 0;$$
$$e\Gamma\Big[(\mu = 1-\gamma)\ \omega,\ \{y_c, 1\}.\begin{pmatrix} \beta + \alpha\,\delta/\mu & \theta + \alpha\,\epsilon/\mu \\ \psi + \delta\,\phi/\mu & \Xi + \phi\,\epsilon/\mu \end{pmatrix}.\{x_c, 1\},$$
$$(\sigma\ /.\ v_{a\mid e} \to 0) + v_c\ (\partial_{v_a}\sigma)\ (\partial_{v_e}\sigma)\Big]$$
$$/.\ \{t_a \to t_c,\ t_e \to t_c,\ b_a \to b_c,\ b_e \to b_c\}\ //\ e\Gamma\text{Collect}\big];$$
$$eR^{+}_{a\_,e\_}\ :=\ e\Gamma\left[1,\ \{y_a, y_e\}.\begin{pmatrix} 1 & 1-t_a \\ 0 & t_a \end{pmatrix}.\{x_a, x_e\},\ v_a + t_a\, v_e\right];$$
$$eR^{-}_{a\_,e\_}\ :=\ eR^{+}_{a,e}\ /.\ t_a \to t_a^{-1};$$

Cheking meta-associativity

In[28]:=
$$e\zeta = e\Gamma\left[\omega,\ \{y_1, y_2, y_3, y_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{x_1, x_2, x_3, x_S\},\ s_1\, v_1 + s_2\, v_2 + s_3\, v_3 + s_S\, v_S\right]$$
$$(e\zeta\ //\ em_{1,2\to 1}\ //\ em_{1,3\to 1}) \equiv (e\zeta\ //\ em_{2,3\to 2}\ //\ em_{1,2\to 1})$$

Out[28]=
$$\left\{\begin{pmatrix} \omega & x_1 & x_2 & x_3 & x_S \\ y_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ y_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ y_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ y_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix},\ s_1\, v_1 + s_2\, v_2 + s_3\, v_3 + s_S\, v_S\right\}$$

Out[29]= True

Strand reversal

In[30]:=
$$dH[a\_][e\Gamma[\omega\_,\ \lambda\_,\ \sigma\_]]\ :=\ \text{Module}\big[\{\alpha, \theta, \phi, \Xi, sa\},$$
$$\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{y_a,x_a}\lambda & \partial_{y_a}\lambda \\ \partial_{x_a}\lambda & \lambda \end{pmatrix}\ /.\ (y\mid x)_a \to 0;$$
$$sa = \partial_{v_a}\sigma;$$
$$e\Gamma\Big[\alpha\,\omega/sa,\ \{y_a, 1\}.\begin{pmatrix} 1/\alpha & \theta/\alpha \\ -\phi/\alpha & (\alpha\,\Xi - \phi\,\theta)/\alpha \end{pmatrix}.\{x_a, 1\},$$
$$\frac{v_a}{sa} + (\sigma\ /.\ \{v_a \to 0\})\Big]\ /.\ \{t_a :\to 1/t_a,\ b_a :\to -b_a\}\ //\ e\Gamma\text{Collect}\big];$$

Verifying the crossings

In[31]:=

$$\left(\mathrm{eR}^{+}_{1,2}\ //\ \mathrm{dH}[1]\right) \equiv \left(\mathrm{eR}^{-}_{1,2}\right)$$
$$\left(\mathrm{eR}^{+}_{1,2}\ //\ \mathrm{dH}[2]\right) \equiv \left(\mathrm{eR}^{-}_{1,2}\right)$$
$$\left(\mathrm{eR}^{-}_{1,2}\ //\ \mathrm{dH}[1]\right) \equiv \left(\mathrm{eR}^{+}_{1,2}\right)$$
$$\left(\mathrm{eR}^{-}_{1,2}\ //\ \mathrm{dH}[2]\right) \equiv \left(\mathrm{eR}^{+}_{1,2}\right)$$

Out[31]= True

Out[32]= True

Out[33]= True

Out[34]= True

Verifying the stitching relations

In[35]:=

$$\zeta = \mathrm{e}\Gamma\left[\omega,\ \{y_b, y_c, y_S\}.\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{x_b, x_c, x_S\},\ s_b v_b + s_c v_c + s_S v_S\right]$$
$$(\zeta\ //\ \mathrm{em}_{b,c\to a}\ //\ \mathrm{dH}[a]) \equiv (\zeta\ //\ \mathrm{dH}[b]\ //\ \mathrm{dH}[c]\ //\ \mathrm{em}_{c,b\to a})$$

Out[35]=

$$\left\{\begin{pmatrix} \omega & x_b & x_c & x_S \\ y_b & \alpha & \beta & \theta \\ y_c & \gamma & \delta & \epsilon \\ y_S & \phi & \psi & \Xi \end{pmatrix},\ s_b v_b + s_c v_c + s_S v_S\right\}$$

Out[36]= True

Strand doubling

In[37]:=

$$\mathrm{q}\Delta[\mathrm{i\_}, \mathrm{j\_}, \mathrm{k\_}][\mathrm{e}\Gamma[\omega\_, \lambda\_, \sigma\_]] := \mathrm{Module}\Big[$$
$$\{\alpha, \theta, \phi, \Xi, \mathrm{si}, \mathrm{M}, \mathrm{ti}, \mu, \nu\},$$
$$\begin{pmatrix} \alpha & \theta \\ \phi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{y_i, x_i}\lambda & \partial_{y_i}\lambda \\ \partial_{x_i}\lambda & \lambda \end{pmatrix}\ /.\ (y \mid x)_i \to 0\ /.\ t_i \to \mathrm{ti};$$
$$\mathrm{si} = \partial_{v_i}\sigma\ /.\ t_i \to \mathrm{ti};\ \mu = -1 + \mathrm{ti};\ \nu = \alpha - \mathrm{si};$$
$$\mathrm{M} = \begin{pmatrix} \frac{-\alpha + \mathrm{ti}\ \mathrm{si} + t_j\ \nu}{\mu} & \frac{(-1+t_j)\ \nu}{\mu} & \frac{(-1+t_j)\ \theta}{\mu} \\ \frac{t_j\ (-1+t_k)\ \nu}{\mu} & \frac{-\mathrm{si} + \mathrm{ti}\ \alpha - t_j\ \nu}{\mu} & \frac{t_j\ (-1+t_k)\ \theta}{\mu} \\ \phi & \phi & \Xi \end{pmatrix};$$
$$\mathrm{e}\Gamma\Big[\omega\ /.\ \{t_i \to t_j\ t_k\},\ \{y_j, y_k, 1\}.\mathrm{M}.\{x_j, x_k, 1\}\ /.\ \{\mathrm{ti} \to t_j\ t_k\},$$
$$\left(\sigma\ /.\ \{v_i \to 0\}\right) + (v_j + v_k)\ \mathrm{si}\ /.\ t_i \mid \mathrm{ti} \to t_j\ t_k\Big]\ //\ \mathrm{e}\Gamma\mathrm{Collect}$$
$$\Big];$$

Verifying the crossings

In[38]:=
```
qΔ[1, 1, 2][eR⁺_{1,3}] ≡ (eR⁺_{2,3} eR⁺_{1,4} // em_{3,4→3})
qΔ[1, 1, 2][eR⁻_{1,3}] ≡ (eR⁻_{1,3} eR⁻_{2,4} // em_{3,4→3})
qΔ[2, 2, 3][eR⁺_{1,2}] ≡ (eR⁺_{1,2} eR⁺_{4,3} // em_{1,4→1})
qΔ[2, 2, 3][eR⁻_{1,2}] ≡ (eR⁻_{1,3} eR⁻_{4,2} // em_{1,4→1})
```

Out[38]= True

Out[39]= True

Out[40]= True

Out[41]= True

Verifying the stitching relations

In[42]:=
```
ζ = eΓ[ω, {y_i, y_j, y_S}.{{α, β, θ}, {γ, δ, ϵ}, {ϕ, ψ, Ξ}}.{x_i, x_j, x_S}, s_i v_i + s_j v_j + s_S v_S]
(ζ // qΔ[i, i1, i2] // qΔ[j, j1, j2] // em_{i1,j1→k1} // em_{i2,j2→k2}) ≡
  (ζ // em_{i,j→k} // qΔ[k, k1, k2])
```

Out[42]= $\left\{\begin{pmatrix} \omega & x_i & x_j & x_S \\ y_i & \alpha & \beta & \theta \\ y_j & \gamma & \delta & \epsilon \\ y_S & \phi & \psi & \Xi \end{pmatrix}, s_i v_i + s_j v_j + s_S v_S\right\}$

Out[43]= True

3. The Lie Algebra G_0

Again first we format G_0

In[44]:=
```
CF[expr_] := expr // Simplify;
𝔼 /: CF[𝔼[ω_, λ_]] := 𝔼[CF[ω], CF[λ]];
𝔼 /: 𝔼[ω1_, λ1_] 𝔼[ω2_, λ2_] := CF@𝔼[ω1 ω2, λ1 + λ2];
𝔼[ω1_, λ1_] ≡ 𝔼[ω2_, λ2_] := CF[ω1 == ω2 ⋀ λ1 == λ2];
```

So for instance

In[48]:=
```
𝔼[ω, Sum[l_{x,y} b_x c_y + q_{x,y} u_x w_y, {x, {i, j}}, {y, {i, j}}]]
```

Out[48]= $\mathbb{E}[\omega, b_i c_i l_{i,i} + b_i c_j l_{i,j} + b_j c_i l_{j,i} + b_j c_j l_{j,j} + u_i w_i q_{i,i} + u_i w_j q_{i,j} + u_j w_i q_{j,i} + u_j w_j q_{j,j}]$

The switching operators

In[49]:=

```
N_{u_i_ c_j_→k_}[𝔼[ω_, λ_]] := CF[
    𝔼[ω, e^{-γ} β u_k + γ c_k + (λ /. c_j | u_i → 0)] /. {γ → ∂_{c_j} λ, β → ∂_{u_i} λ}];
N_{w_i_ c_j_→k_}[𝔼[ω_, λ_]] := CF[
    𝔼[ω, e^γ α w_k + γ c_k + (λ /. c_j | w_i → 0)] /. {γ → ∂_{c_j} λ, α → ∂_{w_i} λ}];
N_{w_i_ u_j_→k_}[𝔼[ω_, λ_]] := CF[
    𝔼[ν ω , -b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (λ /. w_i | u_j → 0)] /. ν → (1 + b_k δ)^{-1}
     /. {α → ∂_{w_i} λ /. u_j → 0, β → ∂_{u_j} λ /. w_i → 0, δ → ∂_{w_i,u_j} λ}];
```

The stitching operation

In[52]:=

```
gm_{i_,j_→k_}[𝔼[ω_, λ_]] := CF[Module[{x},
    (𝔼[ω, λ] // N_{w_i c_j→x} // N_{u_i c_x→x} // N_{w_x u_j→x}) /. {c_i → c_k, w_j → w_k, y_x_ :→ y_k, b_{i|j} → b_k}]]
gR^+_{i_,j_} = 𝔼[1, b_i c_j + b_i^{-1} (e^{b_i} - 1) u_i w_j]; gR^-_{i_,j_} = 𝔼[1, -b_i c_j + b_i^{-1} (e^{-b_i} - 1) u_i w_j];
```

Next we program a map from G_0 to extended Γ-calculus.

In[54]:=

```
G0toΓ[e_] := Module[{A, λ, L, ω, Q, n, II, DD, T, M, σ, i, j},
    ω = e⟦1⟧ /. e^x_ :→ e^{Simplify[x /. b_i_ :→ Log[t_i]]} // Simplify;
    λ = e⟦2⟧;
    A = Union@Cases[λ, (b | c)_a_ :→ a, ∞];
    L = Outer[Factor[∂_{b_#1} ∂_{c_#2} λ] &, A, A];
    Q = Outer[Factor[∂_{u_#1 w_#2} λ] &, A, A];
    n = Length[A];
    II = IdentityMatrix[n];
    DD = DiagonalMatrix[Table[b_i, {i, A}]];
    T = DiagonalMatrix[Table[Product[t_{A⟦i⟧}^{L⟦i,j⟧}, {i, 1, n}], {j, 1, n}]];
    σ = Sum[Product[t_{A⟦i⟧}^{L⟦i,j⟧}, {i, 1, n}] v_{A⟦j⟧}, {j, 1, n}];
    M = T.(II - DD.Q) /. e^x_ :→ e^{Simplify[x /. b_i_ :→ Log[t_i]]} // Simplify;
    eΓ[ω^{-1}, Table[y_i, {i, A}].M.Table[x_i, {i, A}], σ]];
```

Verifying the crossings

In[55]:=

```
eR^+_{i,j} ≡ (gR^+_{i,j} // G0toΓ)
eR^-_{i,j} ≡ (gR^-_{i,j} // G0toΓ)
```

Out[55]= True

Out[56]= True

Verifying the stitching relations

In[57]:=
```
ζ = 𝔼[ω, Sum[l_{x,y} b_x c_y + q_{x,y} u_x w_y, {x, {i, j, S}}, {y, {i, j, S}}]]
(ζ // G0toΓ // em_{i,j→k}) ≡ (ζ // gm_{i,j→k} // G0toΓ)
```

Out[57]= $\mathbb{E}[\omega, b_i c_i l_{i,i} + b_i c_j l_{i,j} + b_i c_S l_{i,S} + b_j c_i l_{j,i} + b_j c_j l_{j,j} + b_j c_S l_{j,S} + b_S c_i l_{S,i} + b_S c_j l_{S,j} + b_S c_S l_{S,S} + u_i w_i q_{i,i} + u_i w_j q_{i,j} + u_i w_S q_{i,S} + u_j w_i q_{j,i} + u_j w_j q_{j,j} + u_j w_S q_{j,S} + u_S w_i q_{S,i} + u_S w_j q_{S,j} + u_S w_S q_{S,S}]$

Out[58]= True

Orientation reversal

In[59]:=
```
gH[a_][e_𝔼] :=
  (e /. {c_a → -c_a, w_a → -w_a, b_a → -b_a, u_a → -u_a}) // N_{u_a c_a→a} // N_{w_a c_a→a} // N_{w_a u_a→a}
```