THE KNUTH RELATIONS

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Abstract. We define three equivalence relations on the set of words using different means. It turns out that they are the same relation.

1. Motivation

Recall the algorithm of row inserting an entry x into a tableau T. We start with the first row of T. If x is bigger than all the entries in the first row of T, we put x at the end of that row. Otherwise let y be the smallest entry that is bigger than x, replace y by x and row insert y into the next row of the tableau. We say that x *bumps* y to the next row. For example,

$$\begin{array}{llll}1&3&5&6\\2&7\end{array} \leftarrow 4, \qquad \begin{array}{llll}1&3&4&6\\2&7\end{array} \leftarrow 5, \qquad \begin{array}{llll}1&3&4&6\\2&5\\7\end{array}.$$

If we define the *row word* $w(T)$ of a tableau T as the juxtaposition of the rows of T from *bottom to top*, then the above row insertion can be expressed as a step-by-step procedure as follows.

$$\begin{aligned}(27|1356)\cdot 4 &\to 2713(564)\\ &\to 271(354)6\\ &\to 27(153)46 \qquad (1)\\ &\to (275)1346\\ &\to 7251346.\end{aligned}$$

Note that every step of the above procedure involves three consecutive entries and we have two main cases. If the first and second entries are both bigger than the third entry, in letters, yzx (the names of the letters remind us of the order of the entries), this suggests that z is not the first entry that is bigger than x and so we want to move x to the left according to the row insertion algorithm, i.e.

$$yzx \ \to\ yxz, \qquad x<y<z.$$

If the second entry is bigger than the third entry but the first entry is less than the third entry, in letters, xzy, this suggests that z is the first entry that is bigger than y and so we want to move z to the left (this is the "bumping" part), i.e.

$$xzy \;\to\; zxy, \qquad x < y < z.$$

These two operations constitute an equivalence relation on the set of words (a word is just a sequence of distinct positive integers), known as Knuth equivalence. Let's make a definition.

Definition 1 (Knuth[1] equivalence)**.** Suppose $x < y < z$. Then two words π, σ *differ by a Knuth relation of the first kind*, written $\pi \overset{1}{\cong} \sigma$, if

$$\pi = x_1 \ldots yzx \ldots x_n \text{ and } \sigma = x_1 \ldots yxz \ldots x_n \text{ or vice versa.} \tag{2}$$

They *differ by a Knuth relation of the second kind*, written $\pi \overset{2}{\cong} \sigma$, if

$$\pi = x_1 \ldots xzy \ldots x_n \text{ and } \sigma = x_1 \ldots zxy \ldots x_n \text{ or vice versa.} \tag{3}$$

Two words are said to be *Knuth equivalent* if they differ by a finite sequence of Knuth relations of the first and second kinds.

Example 1. Consider the symmetric group S_3, in image notation,

$$S_3 = \{123, 213, 132, 321, 231, 312\}.$$

The Knuth equivalence partitions S_3 into equivalence classes

$$\{123\}, \{213, 231\}, \{132, 312\}, \{321\}.$$

For a more complicated example, note that the words 2713564 and 7251346 in (1) are Knuth equivalent.

If we let $P(\pi)$ denote the tableau P in the correspondence $\pi \leftrightarrow (P, Q)$ (recall that P is obtained by successively row inserting the entries of π into the empty tableau), then we obtain the following tableaux for S_3

$$P(123) = \begin{array}{ccc} 1 & 2 & 3 \end{array}, \quad P(213) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \quad P(231) = \begin{array}{cc} 1 & 3 \\ 2 & \end{array},$$

$$P(132) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \quad P(312) = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \quad P(321) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}.$$

Observe that two permutations share the same tableau if and only if they are Knuth equivalent. In Section 3, we'll show that this is not a coincidence. ♣

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[1] named after Donald E. Knuth (1938–), creator of the TEX computer typesetting system

2. Schutzenberger's Jeu De Taquin

Suppose $\mu \subseteq \lambda$ are Ferres diagrams. Then the *skew diagram of shape* λ/μ is the diagram obtained by removing μ from λ. A *skew tableau of shape* λ/μ is a filling of λ/μ by distinct positive integers. If the filling is such that the row entries increase from left to right and the column entries increase from top to bottom, then we have a *standard skew tableau*. The *row word* of a skew tableau is the juxtaposition of the rows of the tableau from bottom to top.

Example 2. Suppose $\lambda = (3, 3, 2, 1)$ and $\mu = (2, 1, 1)$. Then a standard skew tableau of shape λ/μ is

$$\begin{array}{ccc} & & 1 \\ & 3 & 5 \\ & 4 & \\ 2 & & \end{array} .$$

Its row word is 24351.

Note that the word 24351 is not the row word of any tableau (since the column entries don't increase). It is however the row word of a standard skew tableau. In fact, every word is the row word of some standard skew tableau. The choice of the skew tableau is not unique however. The word 24351 is also the row word of

$$\begin{array}{ccccc} & & & & 1 \\ & & & 5 & \\ & & 3 & & \\ & 4 & & & \\ 2 & & & & \end{array} .$$

This is called the *anti-diagonal strip tableau associated with 24351.* ♣

Given a Ferrers diagram λ, an *inner corner of* λ is a box of λ whose removal leaves the Ferrers diagram of a partition. An *outer corner of* λ is a box outside of λ whose addition produces the Ferrers diagram of a partition. So an inner corner of λ has no boxes to the right or below it. For instance, if $\lambda = (5, 4, 2)$, the inner corners are marked with $\bullet$ and the outer corners are marked with $\circ$.

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & & \bullet & \circ \\ \hline & & & \bullet & \circ \\ \cline{1-5} & \bullet & \circ \\ \cline{1-3} \circ \\ \cline{1-1} \end{array} .$$

Now we can define the *Schutzenberger's sliding operation* (or *forward slide*) as follows. Start with a standard skew tableau of shape λ/μ and an *inner corner of* μ, which can be thought of as a hole, or an empty box, and slide the *smaller* of the entries to the *right* and *below* into the empty box. This creates a new empty box in the skew diagram. The process is repeated with this empty box, until the empty box is moved to an *inner corner of* λ, in which case we remove the empty box from the diagram.

The sliding operation is reversible (also known as *backward slide*). We start with an *outer corner of* λ, which can be thought of as an empty box, and slide the *bigger* of the entries to the *left* and *above* into the empty box. This creates a new empty box in the skew diagram. The process is repeated with this empty box, until we reach an *outer corner of* μ, in which case we remove the empty box from the diagram.

Example 3. Consider a standard skew tableau of shape $(5,5,3)/(3,1)$ and an inner corner of μ which is the third box in the first row. The forward sliding procedure is illustrated as follows.

$$\begin{array}{ccccc} & & \square & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & \square & 6 & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & \square & 9 \\ 1 & 3 & 7 & & \end{array}, \quad \begin{array}{ccccc} & & 4 & 5 & 8 \\ & 2 & 6 & 9 & \square \\ 1 & 3 & 7 & & \end{array}.$$

If we start with an outer corner of λ, a backward slide is performed as follows.

$$\begin{array}{ccccc} & & & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & 7 & \square & \end{array}, \quad \begin{array}{ccccc} & & & 5 & 8 \\ & 2 & 4 & 6 & 9 \\ 1 & 3 & \square & 7 & \end{array}, \quad \begin{array}{ccccc} & & & 5 & 8 \\ & 2 & \square & 6 & 9 \\ 1 & 3 & 4 & 7 & \end{array}, \quad \begin{array}{ccccc} & & & 5 & 8 \\ & \square & 2 & 6 & 9 \\ 1 & 3 & 4 & 7 & \end{array}.$$

If we successively apply the Schutzenberger's sliding operation to a standard skew tableau S of shape λ/μ (also known as *jeu the taquin*[2]), pushing the boxes of μ to the inner corners of λ and removing them, we'll obtain a standard tableau in the end. One might wonder whether the final tableau depends on the order in which we remove the boxes of μ. Surprisingly, the answer is no. Moreover, the final tableau we obtain is nothing other than $P(w(S))$. We'll provide an explanation of this fact in Section 3. ♣

3. Main Results

Theorem 1. *Two words π and σ are Knuth equivalent if and only if $P(\pi) = P(\sigma)$.*

Proof. We first show that if π and σ are Knuth equivalent, then $P(\pi) = P(\sigma)$. Since two words are equivalent if they differ by a finite sequence of Knuth relations of the first and second kinds, it suffices to show that $P(\pi) = P(\sigma)$ if $\pi \overset{1}{\cong} \sigma$ or $\pi \overset{2}{\cong} \sigma$.

Suppose π and σ differ by a Knuth relation of the first kind (2). Because there is no change to the entries of π outside yzx, we just need to show that

$$r_x r_z r_y(T) = r_z r_x r_y(T), \quad x < y < z, \tag{4}$$

here by $r_y(T)$ we mean the tableau obtained by row inserting the entry y to a tableau T. We'll prove (4) by induction on the number of rows

[2]teasing game

of T. If T has zero row, i.e. the empty tableau, the left hand side of (4) is given by

$$\emptyset \leftarrow y, \quad y \leftarrow z, \quad y\ z \leftarrow x, \quad \begin{array}{ll} x & z \\ y & \end{array},$$

whereas the right hand side of (4) is

$$\emptyset \leftarrow y, \quad y \leftarrow x, \quad \begin{array}{ll} x & \leftarrow z \\ y & \end{array}, \quad \begin{array}{ll} x & z \\ y & \end{array}.$$

Therefore (4) is true when T has zero row. Now suppose that T has $r > 0$ rows. Let L denote the tableau on the left hand side of (4) and R denote the tableau on the right hand side of (4). Let T', L', R' denote the tableaux obtained from T, L, R by deleting the first row of each tableau, respectively. We first row insert y into T. There are two cases, y is placed at the end of the first row of T or y bumps some entry y' in the first row of T. If y is placed at the end of the first row of T, consider the left hand side of (4). The entry z will also be placed at the end of the first row, after y, and x will bump some entry x' weakly to the left of y. For the right hand side of (4), y will be placed at the end of the first row of T, x will bump some entry x' (same as that on the left hand side) weakly to the left of y and z is placed at the end of the first row after y. It follows that in this case the first rows of L and R are identical and L' is also the same as R' (they are both obtained by row inserting x' into T'). Hence (4) is satisfied.

If y bumps some entry y' in the first row of T, we consider the left hand side of (4). The entry z will either be at the end of the first row of T or bump some entry z' strictly to the right of y (since $z > y$). The entry x will bump some entry x' weakly to the left of y. Note that $x' \le y < y' \le z < z'$. Consideration of the right hand side of (4) will reveal that the first rows of L and R are identical. Moreover,

$$L' = r_{x'} r_{z'} r_{y'}(T'), \quad R' = r_{z'} r_{x'} r_{y'}(T'), \quad x' < y' < z',$$

and so $L' = R'$ by the inductive hypothesis. It follows that $L = R$.

Now suppose that π and σ differ by a Knuth relation of the second kind (3). We want to show that

$$r_y r_z r_x(T) = r_y r_x r_z(T), \quad x < y < z. \tag{5}$$

We also use induction on the number of rows of T. If T is the empty tableau, then (5) is true since

$$\emptyset \leftarrow x, \quad x \leftarrow z, \quad x\ z \leftarrow y, \quad \begin{array}{ll} x & y \\ z & \end{array},$$

$$\emptyset \leftarrow z, \quad z \leftarrow x, \quad \begin{array}{ll} x & \leftarrow y \\ z & \end{array}, \quad \begin{array}{ll} x & y \\ z & \end{array}.$$

Assume T has $r > 0$ rows, we first consider the left hand side of (5). The entry x is either at the end of the first row of T or bumps some entry x'. If x is placed at the end of the first row of T, then z is placed after x and y bumps z to the next row. For the right hand side of (5), z is placed at the end of the first row of T (because $z > x$), x bumps z to the next row (because all the entries in the first row of T are less than x) and y is placed after x. Therefore (5) holds in this case.

The case where x bumps some entry x' in the first row of T is a bit more complicated. We need to decide whether $x' < z$ or $x' > z$. If $x' < z$, then for the left hand side of (5), z either sits at the end of the first row or bumps some entry z' strictly to the right of x, y will bump some entry y' strictly to the right of x and weakly to the left of z. Note that $x < x' < y' \leq z < z'$ and

$$L' = r_{y'} r_{z'} r_{x'}(T'), \quad x' < y' < z'.$$

For the right hand side of (5), z either sits at the end of the first row or bumps some entry z' (same as that on the left hand side) strictly to the right of x' (because $z > x'$), x will bump x' and y will bump y' strictly to the right of x and weakly to the left of z. Therefore the first rows of L and R are equal and

$$R' = r_{y'} r_{x'} r_{z'}(T'), \quad x' < y' < z'.$$

Hence L' and R' are also equal by the inductive hypothesis and (5) is satisfied.

Now if $x' > z$, it follows that x' is the smallest entry that is bigger than z (since all the entries to the left of x' are less than x). We consider the left hand side of (5), x bumps x', z bumps z' right next to x, y bumps z to the next row. We thus obtain

$$L' = r_z r_{z'} r_{x'}(T'), \quad z < x' < z'.$$

For the right hand side of (5), z bumps x', x bumps z to the next row and y bumps z' right next to x. Thus the first rows of L and R are the same and

$$R' = r_{z'} r_z r_{x'}(T'), \quad z < x' < z'.$$

We see that $L' = R'$ because of (4). This completes the first part of the theorem.

For the other part of the theorem, we want to show that if $P(\pi) = P(\sigma)$, then π and σ are Knuth equivalent. Let P denote the tableau $P(\pi) = P(\sigma)$. Let $w(P)$ denote the row word of P (the juxtaposition of rows of P from bottom to top). By transitivity of Knuth equivalence, it suffices to show that π and $w(P)$ are Knuth equivalent (then π and σ are Knuth equivalent since they are both Knuth equivalent to $w(P)$). (As an aside, we have

$$\pi \to P(\pi) \to w(P(\pi)) \overset{K}{\cong} \pi,$$

$$P \to w(P) \to P(w(P)) = P.$$

Thus the operations of obtaining a tableau by successively row inserting the entries of a word and extracting the row word from a tableau can be thought of as inverses of each other, up to Knuth equivalence.) We proceed by induction on the number of entries of π. The case where π has zero entries is trivial. Suppose π has $n > 0$ entries, so that we can write $\pi = \tau x$. Let S denote $P(\tau)$. By induction,

$$\tau \overset{K}{\cong} w(S).$$

Thus

$$\pi = \tau x \overset{K}{\cong} w(S)x.$$

To finish the inductive step, we just need to show that

$$w(S)x \overset{K}{\cong} w(P) = w(S \leftarrow x). \tag{6}$$

(This is essentially true by looking at the example in Section 1. The Knuth relations are defined to express the row insertion algorithm on tableaux in terms of words, suitable for implementation on a computer.) We can prove (6) by induction on the number of rows of S. Again the case where S has zero rows is obvious. Suppose S has $l > 0$ rows and x bumps the entry v_k in the first row of S. Let S' denote the tableau obtained from S by deleting the first row. We have

$$\begin{aligned}
w(S)x &= R_l R_{l-1} \cdots R_1 x \\
&= R_l R_{l-1} \cdots R_2 v_1 \cdots v_k \cdots (v_{m-1} v_m x) \quad (x < v_{m-1} < v_m) \\
&\overset{1}{\cong} R_l R_{l-1} \cdots R_2 v_1 \cdots v_k \cdots (v_{m-2} v_{m-1} x) v_m \quad (x < v_{m-2} < v_{m-1}) \\
&\vdots \\
&\overset{1}{\cong} R_l R_{l-1} \cdots R_2 v_1 \cdots (v_{k-1} v_k x) v_{k+1} \cdots v_m \quad (v_{k-1} < x < v_k) \\
&\overset{2}{\cong} R_l R_{l-1} \cdots R_2 v_1 \cdots (v_{k-2} v_k v_{k-1}) x v_{k+1} \cdots v_m \quad (v_{k-2} < v_{k-1} < v_k) \\
&\vdots \\
&\overset{2}{\cong} R_l R_{l-1} \cdots R_2 v_k v_1 \cdots v_{k-1} x v_{k+1} \cdots v_m \\
&= (w(S') v_k) v_1 \cdots v_{k-1} x v_{k+1} \cdots v_m \\
&= w(S' \leftarrow v_k) v_1 \cdots v_{k-1} x v_{k+1} \cdots v_m \quad \text{(by inductive hypothesis)}
\end{aligned}$$

Note that the last word is precisely the row word of $S \leftarrow x$ (the first row of $S \leftarrow x$ is the same as S with v_k replaced by x and the remaining rows of $S \leftarrow x$ are obtained by row inserting v_k into S'). That completes the proof of the theorem. $\square$

Lemma 1. *Let P and Q be standard skew tableau (we allow an empty box in the tableaux). If P and Q are related by a finite sequence of slides (forward or backward), then $w(P)$ and $w(Q)$ are Knuth equivalent.*

Proof. It suffices to show that $w(P)$ and $w(Q)$ are Knuth equivalent when P and Q are related by a single slide. Observe that a horizontal slide doesn't change the row word of a tableau, hence $w(P) = w(Q)$ and they are Knuth equivalent. Therefore we just need to consider the case where P and Q differ by a vertical slide. Furthermore, we need only focus on the parts of P and Q that are affected by the slide, as illustrated in the next figure.

$$P = \begin{array}{|c|c|c|c|c|c|c|} \cline{2-7} \multicolumn{1}{c|}{} & R_l & \cdots & & \cdots & R_r & \cdots \\ \hline \cdots & S_l & \cdots & x & \cdots & S_r & \multicolumn{1}{c}{} \\ \cline{1-6} \end{array},$$

$$Q = \begin{array}{|c|c|c|c|c|c|c|} \cline{2-7} \multicolumn{1}{c|}{} & R_l & \cdots & x & \cdots & R_r & \cdots \\ \hline \cdots & S_l & \cdots & & \cdots & S_r & \multicolumn{1}{c}{} \\ \cline{1-6} \end{array}.$$

Here the entry x slides into the empty box. We let R_l, S_l denote the upper and lower rows to the left of x respectively and R_r, S_r denote the upper and lower rows to the right of x respectively. We want to show that the row words for these portions of P and Q are Knuth equivalent. We proceed by induction on the sizes of R_l, S_l, R_r, S_r. If all of them are zero, then the result is clearly true. Otherwise let $|R|$ denote the number of entries of row R. If $|R_r| > |S_r|$ (or $|S_l| > |R_l|$), then we can just remove the last (first) entry of R_r (S_l) and the result is satisfied by the inductive hypothesis. Therefore the only case we need to consider is when $|R_l| = |S_l|$ and $|R_r| = |S_r|$. Suppose

$$\begin{aligned} R_l &= x_1 \cdots x_j, \quad & R_r &= y_1 \cdots y_k, \\ S_l &= z_1 \cdots z_j, \quad & S_r &= w_1 \cdots w_k. \end{aligned}$$

The row word of P is

$$w(P) = z_1 \cdots z_j x w_1 \cdots w_k x_1 \cdots x_j y_1 \cdots y_k$$

and the row word of Q is

$$w(Q) = z_1 \cdots z_j w_1 \cdots w_k x_1 \cdots x_j x y_1 \cdots y_k$$

and we want to show that they are Knuth equivalent. Note that since P is a standard skew tableau, we have

$$x_1 < z_1 < \cdots < z_j < x < w_1 < \cdots < w_k.$$

We want to move x_1 to the left. Apply the Knuth relation of the first kind (2) to $w(Q)$ repeatedly, starting from the triple $w_{k-1}w_kx_1$ we obtain

$$\begin{aligned} w(Q) &= z_1 \cdots z_j w_1 \cdots (w_{k-1} w_k x_1) x_2 \cdots x_j x y_1 \cdots y_k \quad (x_1 < w_{k-1} < w_k) \\ &\overset{1}{\cong} z_1 \cdots z_j w_1 \cdots (w_{k-2} w_{k-1} x_1) w_k x_2 \cdots x_j x y_1 \cdots y_k \quad (x_1 < w_{k-2} < w_{k-1}) \\ &\vdots \\ &\overset{1}{\cong} z_1 x_1 (z_2 \cdots z_j w_1 \cdots w_k x_2 \cdots x_j x y_1 \cdots y_k). \end{aligned}$$

By the inductive hypothesis for $|R_l|-1$, $|S_l|-1$,

$$w(Q) \overset{K}{\cong} z_1 x_1 (z_2 \cdots z_j x w_1 \cdots w_k x_2 \cdots x_j y_1 \cdots y_k).$$

Now we move x_1 to the right, again by using the Knuth relation of the first kind (2),

$$\begin{aligned} w(Q) &\overset{K}{\cong} (z_1 x_1 z_2) z_3 \cdots z_j x w_1 \cdots w_k x_2 \cdots x_j y_1 \cdots y_k \quad (x_1 < z_1 < z_2) \\ &\overset{1}{\cong} z_1 (z_2 x_1 z_3) \cdots z_j x w_1 \cdots w_k x_2 \cdots x_j y_1 \cdots y_k \quad (x_1 < z_2 < z_3) \\ &\vdots \\ &\overset{1}{\cong} z_1 z_2 \cdots z_j x w_1 \cdots w_k x_1 x_2 \cdots x_j y_1 \cdots y_k \\ &= w(P) \end{aligned}$$

and that completes the proof. □

Lemma 2. *Let S be a standard skew tableau of shape λ/μ. If $\widetilde{S}$ is a standard tableau obtained from S by Jeu de Taquin as described in Section 2, then $\widetilde{S}$ is unique. Moreover, $\widetilde{S}$ is precisely $P(w(S))$.*

Proof. We know that $w(S)$ and $w(\widetilde{S})$ are Knuth equivalent from Lemma 1. From Theorem 1, it follows that $P(w(S)) = P(w(\widetilde{S}))$. Since $\widetilde{S}$ is standard, $P(w(\widetilde{S}))$ is the same as $\widetilde{S}$ and the result follows. □

Theorem 2. *Let S and T be standard skew tableaux. Then $w(S)$ and $w(T)$ are Knuth equivalent if and only if S and T are related by a finite sequence of slides.*

Proof. If S and T are related by a finite sequence of slides, then their words are Knuth equivalent by Lemma 1. Conversely, if $w(S)$ and $w(T)$ are Knuth equivalent, then $P(w(S)) = P(w(T))$ by Theorem 1. From Lemma 2, it follows that $\widetilde{S} = \widetilde{T}$. Hence S and T are related by a finite sequence of slides (each one is related to its final tableau by a finite sequence of slides). □

4. Summary

We have defined three equivalence relations on the set of words. The first relation is the Knuth equivalence. For the second relation, we consider the standard tableau $P(\pi)$ associated with a word π obtained by the row insertion algorithm and say that two words are equivalent if and only if they have the same associated tableaux. For the third relation, we associate with every word π a standard skew tableau whose row word is π (the choice is not unique, Example 2) and declare two words are equivalent if and only if their corresponding skew tableaux are related by a sequence of slides. We have shown that these three equivalence relations are the same (Theorem 1 and Theorem 2).

References

[1] William Fulton. *Young Tableaux, with Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.

[2] Bruce E. Sagan. *The Symmetric Group, Representations, Combinatorial Algorithms, and Symmetric Functions.* Springer, second edition, 2001.