INTRODUCTION TO SPECTRAL SEQUENCES

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1. Introduction

In homological algebra and algebraic topology, a spectral sequence is a means of computing homology groups by taking successive approximations. Roughly speaking, a spectral sequence is a sequence $(E^r, d^r)_{r\geq 1}$ of bicomplexes, which is can think of as a "book" with infinitely many "pages". Every time we go to the next page, we get a bit closer to the homology of the complex we want to compute. Spectral sequences were invented in the 1940s, independently, by J. Leray and R. C. Lyndon. Leray was a German prisoner of war from 1940 through 1945, during World War II.

This notes aims to give a brief introduction to spectral sequences from a purely homological algebra point of view. Specifically, we only describe the set-up of spectral sequences, without going into their many exciting applications in topology and geometry. For this notes, we closely follow the treatment found in [1]. We only assume very basic homological algebra facts. In particular, we assume the readers are comfortable with the notions of complexes, homology, exact sequence in homologies. Some technical proofs which are not very essential to understanding the flow of ideas are referenced in [1].

The organization of the notes goes as follows. In Section 2, we introduce our objects of study: bicomplexes and their total complexes. Our main goal is to compute the homology of the total complex of a complex. In Section 3, we describe the essential concept of a filtration of a complex and how to go from a filtration of a complex to an exact couple, which then gives rise to spectral sequence, by a recursive process. In Section 4, we discuss spectral sequences and show that a spectral sequence arising from a filtration of a complex "converges" to the homology of the complex. Here by convergence, we mean only up to extension, i.e. there exists a short exact sequence

$$0 \longrightarrow \Phi^{p-1}H_n \longrightarrow \Phi^p H_n \longrightarrow E^{\infty}_{p,q} \longrightarrow 0.$$

In Section 5, we apply our general discussion to the case of the total complex of a bicomplex and describe a concrete way to compute the first two pages of the spectral sequence using iterated homologies. Finally, in Section 6, we illustrate how to use the machinery we have developed to show that the definition of the Tor functor is independent of the variable resolved.

In this notes R will denote a commutative ring with 1. It follows that left R-modules are the same as right R-modules, and we just call them R-modules.

2. Bicomplexes

Definition 2.1. A ***graded module*** is an indexed family $M = (M_p)_{p\in\mathbb{Z}}$ of R-modules.

Definition 2.2. Let M, N be graded modules and let $a \in \mathbb{Z}$. A ***graded map of degree*** a, denoted by $f\colon M \to N$, is a family of R-module homomorphisms $f = (f_p\colon M_p \to N_{p+a})_{p\in\mathbb{Z}}$. The ***degree*** of f is a, and we denote it by $\deg(f) = a$.

Let M and M' be graded modules, then M' is a ***submodule*** of M, denoted $M' \subseteq M$, if each M'_p is a submodule of M_p for all $p \in \mathbb{Z}$. If M' is a submodule of M, then we can form the ***quotient*** $M/M' = (M_p/M'_p)_{p\in\mathbb{Z}}$. If $f\colon M \to N$ is a graded map of degree a, then the ***kernel*** of f is the graded module $\ker f = (\ker f_p)_{p\in\mathbb{Z}} \subseteq M$ and the ***image*** of f is the graded module $\operatorname{im} f = (\operatorname{im} f_{p-a})_{p\in\mathbb{Z}} \subseteq N$. A sequence of graded maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is ***exact*** if $\operatorname{im} f = \ker g$, i.e. if $\operatorname{im} f_{p-a} = \ker g_p$ for all $p \in \mathbb{Z}$.

Definition 2.3. A ***complex*** (or ***chain complex***) is an ordered pair (C, d), where C is a graded module and d is a graded map of degree -1 that satisfies $dd = 0$. The map d is called the ***differential***.

$$C\colon\ \cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

The condition $dd = 0$ implies that $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for all $n \in \mathbb{Z}$. The ***n*th homology module of** (C, d) is defined as

$$H_n(C) = \ker d_n / \operatorname{im} d_{n+1}.$$

Definition 2.4. A ***bigraded module*** is a doubly indexed family $M = (M_{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ of R-modules.

Definition 2.5. Let M, N be bigraded modules and let $(a, b) \in \mathbb{Z}\times\mathbb{Z}$. A ***bigraded map of degree*** (a, b), denoted by $f\colon M \to N$, is a family of R-module homomorphisms $f = (f_{p,q}\colon M_{p,q} \to N_{p+a,q+b})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. The ***bidegree*** of f is (a, b), and we denote it by $\deg(f) = (a, b)$.

Let M and M' be bigraded modules, then M' is a ***submodule*** of M, denoted $M' \subseteq M$, if each $M'_{p,q}$ is a submodule of $M_{p,q}$ for all $(p, q) \in \mathbb{Z}\times\mathbb{Z}$. If M' is a submodule of M, then we can form the ***quotient*** $M/M' = (M_{p,q}/M'_{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$. If $f\colon M \to N$ is a bigraded map of bidegree (a, b), then the ***kernel*** of f is the bigraded module $\ker f = (\ker f_{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} \subseteq M$ and the ***image*** of f is the bigraded module $\operatorname{im} f = (\operatorname{im} f_{p-a,q-b})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}} \subseteq N$. A sequence of bigraded maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is ***exact*** if $\operatorname{im} f = \ker g$, i.e. if $\operatorname{im} f_{p-a,q-b} = \ker g_{p,q}$ for all $(p, q) \in \mathbb{Z}\times\mathbb{Z}$.

Definition 2.6. A ***differential bigraded module*** is an ordered pair (M, d), where M is a bigraded module and $d\colon M \to M$ is a bigraded map with $dd = 0$. The map d is called the ***differential***.

If (M, d) is a bigraded module, where d has bidegree (a, b), then its ***homology*** $H(M, d)$ is the bigraded module whose (p, q) term is

$$H(M, d)_{p,q} = \frac{\ker d_{p,q}}{\operatorname{im} d_{p-a,q-b}}.$$

Definition 2.7. A ***bicomplex*** (or ***double complex***) is an ordered triple (M, d', d''), where $M = (M_{p,q})$ is a bigraded module, $d', d''\colon M \to M$ are bigraded maps of bidegree $(-1, 0)$ and $(0, -1)$, respectively that satisfy

(i) $d'd' = 0$,
(ii) $d''d'' = 0$,
(iii) $d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0$

Thus we can visualize the bicomplex (M, d', d'') as follows. Put $M_{p,q}$ at the lattice point (p, q) of $\mathbb{Z}\times\mathbb{Z}$. The map d' points to the left and the map d'' points down. The first condition says that each row forms a complex with differentials d' and the second condition says that each column forms a complex with differentials d''. The third condition says that each square of the lattice anticommutes.

Definition 2.8. If (M, d', d'') is a bicomplex, then its ***total complex***, denoted by $\mathrm{Tot}(M)$, is the complex with nth term

$$\mathrm{Tot}(M)_n = \bigoplus_{p+q=n} M_{p,q}$$

and with differentials $D_n\colon \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$ given by $D_n = \sum_{p+q=n}(d'_{p,q} + d''_{p,q})$.

$$\begin{tikzcd}
M_{p-2,q+1} & M_{p-1,q+1} \arrow[l, "d'_{p-1,q+1}"'] \arrow[d, "d''_{p-1,q+1}"'] & \\
& M_{p-1,q} & M_{p,q} \arrow[l, "d'_{p,q}"'] \arrow[d, "d''_{p,q}"] \\
& & M_{p,q-1}
\end{tikzcd}$$

Lemma 2.9. *If (M, d', d'') is a bicomplex, then $(\mathrm{Tot}(M), D)$ is a complex.*

Proof. We need to check that indeed $D_n\colon \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$ and $D_{n-1}D_n = 0$. This is where condition (iii) in definition 2.7 comes into play. The readers are encouraged to draw the picture and see that directly. □

3. Filtrations and Exact Couples

Definition 3.1. A ***filtration*** of an R-module M is a family $(M_p)_{p\in\mathbb{Z}}$ of submodules of M such that

$$\cdots \subseteq M_{p-1} \subseteq M_p \subseteq M_{p+1} \subseteq \cdots .$$

For each $p \in \mathbb{Z}$, the module M_p/M_{p-1} is called a ***factor module*** of the filtration.

Definition 3.2. A ***filtration*** of a complex (C, d) is a sequence of graded submodules $(F^pC)_{p\in\mathbb{Z}}$ of C such that

$$\cdots \subseteq F^{p-1}C \subseteq F^pC \subseteq F^{p+1}C \subseteq \cdots$$

that satisfies

$$d\colon (F^pC)_n \to (F^pC)_{n-1}$$

for all n, p. To simplify notation, we denote $(F^pC)_n$ by F^p_n. Note that the following diagram commutes.

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F^{p-1}_{n+1} \arrow[r] \arrow[d] & F^p_{n+1} \arrow[r] \arrow[d] & F^{p+1}_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^{p-1}_{n} \arrow[r] \arrow[d] & F^p_{n} \arrow[r] \arrow[d] & F^{p+1}_{n} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^{p-1}_{n-1} \arrow[r] \arrow[d] & F^p_{n-1} \arrow[r] \arrow[d] & F^{p+1}_{n-1} \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

The horizontal maps are inclusions and the vertical maps are restrictions of d to F^p_n. The nth row form a filtration of C_n.

Definition 3.3. Let (M, d', d'') be a bicomplex. The ***first filtration*** (or ***vertical filtration***) of $\mathrm{Tot}(M)$ is given by

$$\begin{aligned}({}^{\mathrm{I}}F^p\mathrm{Tot}(M))_n &= \bigoplus_{i\le p} M_{i,n-i}\\ &= \cdots \oplus M_{p-2,q+2} \oplus M_{p-1,q+1} \oplus M_{p,q}.\end{aligned}$$

Here we use the convention that $n = p + q$. Thus we have

$$\cdots \subseteq ({}^{\mathrm{I}}F^{p-1}\mathrm{Tot}(M))_n \subseteq ({}^{\mathrm{I}}F^{p}\mathrm{Tot}(M))_n \subseteq ({}^{\mathrm{I}}F^{p+1}\mathrm{Tot}(M))_n \subseteq \cdots$$

a filtration of $\mathrm{Tot}(M)_n$. Pictorially, $({}^{\mathrm{I}}F^p\mathrm{Tot}(M))_n$ is the direct sum of the terms on the line $x + y = n$ which lie to the left of the vertical line $x = p$.

Definition 3.4. Let (M, d', d'') is a bicomplex. The ***second filtration*** of $\mathrm{Tot}(M)$ (or ***horizontal filtration***) is given by

$$\begin{aligned}({}^{\mathrm{II}}F^p\mathrm{Tot}(M))_n &= \bigoplus_{j\le p} M_{n-j,j}\\ &= \cdots \oplus M_{q-1,p-2} \oplus M_{q+1,p-1} \oplus M_{q,p},\end{aligned}$$

where we again use the convention $p + q = n$. We have

$$\cdots \subseteq ({}^{\mathrm{II}}F^{p-1}\mathrm{Tot}(M))_n \subseteq ({}^{\mathrm{II}}F^{p}\mathrm{Tot}(M))_n \subseteq ({}^{\mathrm{II}}F^{p+1}\mathrm{Tot}(M))_n \subseteq \cdots$$

a filtration of $\mathrm{Tot}(M)_n$. Pictorially, $({}^{\mathrm{II}}F^p\mathrm{Tot}(M))_n$ is the direct sum of the terms on the line $x + y = n$ which lie below the horizontal line $y = p$.

Exercise 3.5. Check that the above definitions indeed give us filtrations of $\mathrm{Tot}(M)$, i.e. check that $d\colon (F^p\mathrm{Tot}(M))_n \to (F^p\mathrm{Tot}(M))_{n-1}$.

To see the relationship between the first filtration and the second filtration, we introduce the concept of the transpose of a bicomplex.

Definition 3.6. Let (M, d', d'') be a bicomplex. The ***transpose*** of (M, d, d'') is the bicomplex (M^t, δ', δ''), where $M^t_{p,q} = M_{q,p}$, $\delta'_{p,q} = d''_{q,p}$ and $\delta''_{p,q} = d'_{q,p}$.

Lemma 3.7. *Let (M, d', d'') be a bicomplex. Then*
(i) $\mathrm{Tot}(M) = \mathrm{Tot}(M^t)$,
(ii) *the second filtration of* $\mathrm{Tot}(M)$ *is equal to the first filtration of* $\mathrm{Tot}(M^t)$, *i.e.*

$${}^{\mathrm{II}}F^p\mathrm{Tot}(M)_n = {}^{\mathrm{I}}F^p\mathrm{Tot}(M^t)_n.$$

Proof. The proof consists of manipulating the indices. We have

$$\mathrm{Tot}(M^t)_n = \bigoplus_{p+q=n} M^t_{p,q} = \bigoplus_{p+q=n} M_{q,p} = \mathrm{Tot}(M)_n.$$

Also,

$${}^{\mathrm{II}}F^p\mathrm{Tot}(M)_n = \bigoplus_{j\le p} M_{n-j,j} = \bigoplus_{j\le p} M^t_{j,n-j} = {}^{\mathrm{I}}F^p\mathrm{Tot}(M^t)_n,$$

as required. □

Definition 3.8. An ***exact couple*** is a 5-tuple $(D, E, \alpha, \beta, \gamma)$, where D and E are bigraded modules, α, β, γ are bigraded maps with bidegrees (a, a'), (b, b') and (c, c'), respectively, and there is exactness at each vertex: $\ker\alpha = \mathrm{im}\,\gamma$, $\ker\beta = \mathrm{im}\,\alpha$, and $\ker\gamma = \mathrm{im}\,\beta$.

Thus an exact couple is equivalent to a collection of exact sequences: for each $(p, q) \in \mathbb{Z}\times\mathbb{Z}$ we have an exact sequence:

$$\cdots \longrightarrow A_{p,q} \longrightarrow B_{p+a,q+a'} \longrightarrow C_{p+a+b,q+a'+b'} \longrightarrow A_{p+a+b+c,q+a'+b'+c'} \longrightarrow \cdots$$

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[rr, "\alpha\ (1,-1)"] & & D \arrow[dl, "\beta\ (0,0)"] \\ & E \arrow[ul, "\gamma\ (-1,0)"] & \end{tikzcd}$$

Proposition 3.9. *Every filtration $(F^pC)_{p\in\mathbb{Z}}$ of a complex (C, d) determines an exact couple whose bigraded maps have the displayed bidegrees.*

Proof. To simplify notation, we write F^pC as F^p. For each $p \in \mathbb{Z}$, we have the following short exact sequence of complexes:

$$0 \longrightarrow F^{p-1} \longrightarrow F^p \longrightarrow F^p/F^{p-1} \longrightarrow 0,$$

or in expanded form.

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & F^{p-1}_{n+1} \arrow[r] \arrow[d] & F^p_{n+1} \arrow[r] \arrow[d] & F^p_{n+1}/F^{p-1}_{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^{p-1}_{n} \arrow[r] \arrow[d] & F^p_{n} \arrow[r] \arrow[d] & F^p_{n}/F^{p-1}_{n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^{p-1}_{n-1} \arrow[r] \arrow[d] & F^p_{n-1} \arrow[r] \arrow[d] & F^p_{n-1}/F^{p-1}_{n-1} \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

This induces a long exact sequence in homologies (see [1, **Theorem 6.10**]).

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(F^{p-1}) \arrow[r] & H_{n+1}(F^p) \arrow[r] & H_{n+1}(F^p/F^{p-1}) \arrow[dll, "\delta"'] & \\ & H_n(F^{p-1}) \arrow[r] & H_n(F^p) \arrow[r] & H_n(F^p/F^{p-1}) \arrow[dll, "\delta"'] & \\ & H_{n-1}(F^{p-1}) \arrow[r] & H_{n-1}(F^p) \arrow[r] & H_{n-1}(F^p/F^{p-1}) \arrow[r] & \cdots \end{tikzcd}$$

Now for each $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, put

$$D_{p,q} = H_{p+q}(F^p), \quad E_{p,q} = H_{p+q}(F^p/F^{p-1}).$$

We then obtain a long exact sequence.

$$\cdots \longrightarrow D_{p-1,q+2} \longrightarrow D_{p,q+1} \longrightarrow E_{p,q+1} \longrightarrow D_{p-1,q+1} \longrightarrow D_{p,q} \longrightarrow$$
$$\longrightarrow E_{p,q} \longrightarrow D_{p-1,q} \longrightarrow D_{p,q-1} \longrightarrow \cdots,$$

which is equivalent to the exact couple

$$\begin{tikzcd} D \arrow[rr, "\alpha\ (1,-1)"] & & D \arrow[dl, "\beta\ (0,0)"] \\ & E \arrow[ul, "\gamma\ (-1,0)"] & \end{tikzcd}$$

□

Proposition 3.10. *If $(D, E, \alpha, \beta, \gamma)$ is an exact couple, where α, β, γ have bidegrees (a, a'), (b, b'), (c, c'), respectively, then $d^1 = \beta\gamma$ is a differential $d^1 \colon E \to E$, and there is an exact couple $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$, called the* **derived couple**, *with $E^2 = H(E, d^1)$, and whose bigraded maps have the displayed bidegrees.*

$$\begin{tikzcd} D^2 \arrow[rr, "\alpha^2\ (a,a')"] & & D^2 \arrow[dl, "\beta^2\ (b-a,b'-a')"] \\ & E^2 \arrow[ul, "\gamma^2\ (c,c')"] & \end{tikzcd}$$

Proof. We refer the readers to [1, **Proposition 10.9**] for a proof. For our purpose, we just need to know the definitions of the terms in the derived couple. We have $E^2 = H(E, d^1)$ and $D^2 = \operatorname{im} \alpha \subseteq D$. Now the bigraded map $\alpha^2 \colon D^2 \to D^2$ is the restriction of α to D^2; the bigraded map $\beta^2 \colon D^2 \to E^2 = H(E, d^1)$ is given by

$$\beta^2(y) = [\beta\alpha^{-1}(y)] \quad \text{for } y \in D^2_{p,q};$$

and the bigraded map $\gamma^2 \colon E^2 \to D^2$ is given by

$$\gamma^2([z]) = \gamma z$$

for $[z] \in E^2_{p,q}$. □

The above procedure can be iterated, i.e. given $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$, we can obtain an exact couple $(D^3, E^3, \alpha^3, \beta^3, \gamma^3)$, with $E^3 = H(E^2, d^2)$. More generally, we define the $(r+1)$***st derived couple*** $(D^{r+1}, E^{r+1}, \alpha^{r+1}, \beta^{r+1}, \gamma^{r+1})$ to be the derived couple of $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$. For consistency of notation, we also call $(D, E, \alpha, \beta, \gamma)$ the ***first derived couple***.

Now we specialize the above proposition to the exact couple arising from a filtration to obtain the following corollary.

Corollary 3.11. *Let $(D, E, \alpha, \beta, \gamma)$ be the exact couple arising from a filtration (F^p) of a complex (C, d), so α, β, γ have bidegrees $(1, -1)$, $(0, 0)$, $(-1, 0)$ respectively. Then the rth derived couple $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ has the following properties:*

(i) *the bigraded maps α^r, β^r, γ^r have bidegrees $(1, -1)$, $(1 - r, r - 1)$, $(-1, 0)$, respectively;*
(ii) *the differential d^r has bidegree $(-r, r - 1)$, and it is induced by $\beta\alpha^{-r+1}\gamma$;*
(iii) $E^{r+1}_{p,q} = \ker f^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$;
(iv) $D^r_{p,q} = \operatorname{im} (\alpha_{p-1,q+1})(\alpha_{p-2,q+2}) \cdots (\alpha_{p-r+1,q+r-1})$; *in particular, we have*

$$D^r_{p,q} = \operatorname{im}\ \left[(j^{p-1} j^{p-2} \cdots j^{p-r+1})_* \colon H_n(F^{p-r+1}) \to H_n(F^p)\right],$$

where j^p denotes the inclusion $F^p \hookrightarrow F^{p+1}$.

Proof. When we go to the next level, the bidegrees of α and γ stay the same, whereas the bidegree of β decreases by $(1,-1)$. Therefore the bidegree of α^r is still $(-1,1)$, the bidegree of γ^r is still $(-1,0)$, whereas the bidegree of β^r is $(1-r, r-1)$. For the differential d^r, note that

$$d^r = \beta^{r-1}\gamma^{r-1} = \beta^{r-2}(\alpha^{r-2})^{-1}\gamma^{r-2} = \beta^{r-3}(\alpha^{r-3})^{-1}\alpha^{-1}\gamma = \cdots = \beta\alpha^{-r+1}\gamma.$$

Hence the degree of d^r is $(-r+1, r-1) + (-1,0) = (-r, r-1)$. It follows that

$$E^{r+1}_{p,q} = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$

Now for the last statement, recall that $D^2_{p,q} = \alpha_{p-1,q+1}D_{p-1,q+1}$. Thus

$$D^3_{p,q} = \alpha_{p-1,q+1}D^2_{p-1,q+1} = \alpha_{p-1,q+1}\alpha_{p-2,q+2}D_{p-2,q+2}.$$

It follows that

$$D^r_{p,q} = \operatorname{im}\alpha_{p-1,q+1}\alpha_{p-2,q+2}\cdots\alpha_{p-r+1,q+r-1}.$$

Now from the exact sequence in **Proposition 3.9**, we know that $D_{p,q} = H_{p+q}(F^p)$ and $\alpha_{p,q}\colon H_{p+q}(F^p) \to H_{p+q}(F^{p+1})$ is just the induced map from the inclusion $j^p\colon F^p \to F^{p+1}$. Therefore,

$$D^r_{p,q} = \operatorname{im}(j^{p-1}j^{p-2}\cdots j^{p-r+1})_*,$$

as claimed. □

4. Spectral Sequences

Definition 4.1. A ***spectral sequence*** is a sequence $(E^r, d^r)_{r\geq 1}$ of differential bigraded modules such that $E^{r+1} = H(E^r, d^r)$ for all r.

We have proved the following theorem.

Theorem 4.2. *Every filtration of a complex produces a spectral sequence.*

Proof. Combine **Proposition 3.9** and **Proposition 3.10** □

Definition 4.3. A filtration (F^pC) of a complex (C,d) is ***bounded*** if for each n, there exist integers $s = s(n)$ and $t = t(n)$ such that

$$(F^sC)_n = \{0\} \quad \textit{and} \quad (F^tC)_n = C_n.$$

Lemma 4.4. *Let (E^r, d^r) be the spectral sequence arising from a bounded filtration of a complex. Then for each $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ there exists an integer r (depending on p, q) such that $E^s_{p,q} = E^r_{p,q}$ for all $s > r$.*

Proof. For a fixed position (p,q), let $n = p+q$. Recall that $E_{p,q} = H_n(F^p/F^{p-1})$ (see **Proposition 3.9**). For $p > t(n)$, $F^p_n/F^{p-1}_n = 0$. Therefore

$$E_{p,q} = H_n(F^p/F^{p-1}) \subseteq F^p_n/F^{p-1}_n = 0.$$

So $E_{p,q} = 0$. It follows that $E^r_{p,q} = 0$ for all $r \geq 1$ and $p > t(n)$ since $E^r_{p,q} = H_{p,q}(E^{r-1})$. If $p < s(n)$, then $F^p_n = 0$ and so $F^p_n/F^{p-1}_n = 0$. Thus $E^r_{p,q} = 0$ for all $r \geq 1$ and $p < s(n)$.

Now choose r big enough so that $p+r > t(n+1)$ and $p-r < s(n-1)$. Recall that the differential d^r has bidegree $(-r, r-1)$. Therefore

$$E^{r+1}_{p,q} = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}} = \frac{E^r_{p,q}}{\{0\}} = E^r_{p,q}$$

because $E^r_{p-r,q+r-1} = 0$ and $E^r_{p+r,q-r+1} = 0$. □

Definition 4.5. For the spectral sequence arising from a bounded filtration of a complex, we define

$$E^{\infty}_{p,q} \colon = E^{r}_{p,q},$$

where p, q, r are as in the above lemma.

Given a filtration $(F^pC)_{p\in\mathbb{Z}}$ of a complex (C,d) with inclusions $i^p_n \colon (F^pC)_n \to C_n$, then $i^p_* \colon H(F^p) \to H(C)$. Since $F^p \subseteq F^{p+1}$, we have $\operatorname{im} i^p_* \subseteq \operatorname{im} i^{p+1}_*$; that is, $(\operatorname{im} i^p_*)$ is a filtration of $H(C)$.

Definition 4.6. Let $(F^pC)_{p\in\mathbb{Z}}$ is a filtration of a complex (C,d) and $i^p \colon F^p \to C$ are inclusions. Define

$$\Phi^p H_n(C) = \operatorname{im} i^p_*, \quad i^p_* \colon H_n(F^p) \to H_n(C).$$

We call $(\Phi^p H_n(C))_{p\in\mathbb{Z}}$ the ***induced filtration*** of $H_n(C)$.

Lemma 4.7. *If $(F^pC)_{p\in\mathbb{Z}}$ is a bounded filtration of a complex (C,d), then the induced filtration $(\Phi^p H(C))_{p\in\mathbb{Z}}$ on homology is also bounded with the same bound. More precisely, if for $n \in \mathbb{Z}$ there exist s, t depending on n such that*

$$\{0\} = F^s C_n \subseteq F^{s+1} C_n \subseteq \cdots \subseteq F^t C_n = C_n,$$

then

$$\{0\} = \Phi^s H_n(C) \subseteq \Phi^{s+1} H_n(C) \subseteq \cdots \subseteq \Phi^t H_n(C) = H_n(C).$$

Proof. Since $F^s_n = 0$, $H_n(F^s) = 0$ and so $\operatorname{im} i^s_* = 0$. Since $F^t_n = C_n$, $i^t \colon F^t C_n \to C_n$ is just the identity map. Thus $i^t_* \colon H_n(F^t) \to H_n(C)$ is also the identity map. Therefore $i^t_*(H_n(F^t)) = H_n(C)$. □

Definition 4.8. A spectral sequence $(E^r, d^r)_{r\geq 1}$ ***converges*** to a graded module H, denoted by

$$E^2_{p,q} \underset{p}{\Rightarrow} H_n,$$

if there is some *bounded* filtration $(\Phi^p H)_{p\in\mathbb{Z}}$ of H such that

$$E^{\infty}_{p,q} \cong \Phi^p H_n / \Phi^{p-1} H_n$$

for all n, where we use the convention $n = p + q$.

Theorem 4.9. *Let $(F^pC)_{p\in\mathbb{Z}}$ be a bounded filtration of a complex (C,d) (so that the induced filtration $(\Phi^p H)$ is bounded with the same bound) and $(E^r, d^r)_{r\geq 1}$ be the corresponding spectral sequence. Then*

$$E^2_{p,q} \underset{p}{\Rightarrow} H_n(C).$$

Proof. Fix a position $(p,q) \in \mathbb{Z}^2$, let $n = p+q$. Consider the exact sequence obtained from the rth derived couple:

$$D^r_{p+r-2,\#} \xrightarrow{\alpha^r} D^r_{p+r-1,\#} \xrightarrow{\beta^r} E^r_{p,q} \xrightarrow{\gamma^r} D^r_{p-1,q}.$$

Recall that (see **Corollary 3.11**)

$$D^r_{p,q} = \operatorname{im} \left[(j^{p-1} j^{p-2} \cdots j^{p-r+1})_* \colon H_n(F^{p-r+1}) \to H_n(F^p) \right],$$

where j^p denotes the inclusion $F^p \hookrightarrow F^{p+1}$. Replacing p by $p+r-1$ and $p+r-2$ we obtain

$$D^r_{p+r-1,\#} = \operatorname{im} \left[(j^{p+r-2} j^{p+r-3} \cdots j^{p})_* \colon H_n(F^{p}) \to H_n(F^{p+r-1}) \right],$$

and

$$D^r_{p+r-2,\#} = \operatorname{im} \left[(j^{p+r-3} j^{p+r-4} \cdots j^{p-1})_* \colon H_n(F^{p-1}) \to H_n(F^{p+r-2}) \right].$$

For large r such that $p + r - 2 > t(n)$, we have $F^{p+r-1}_n = F^{p+r-2}_n = C_n$ and the composite $j^{p+r-2} \cdots j^p$ of inclusions is just the inclusion $i^p \colon F^p_n \to C_n$. Therefore, $D^r_{p+r-1,\#} = \operatorname{im} i^p_* = \Phi^p H_n$. Similarly, $D^r_{p+r-2,\#} = \Phi^{p-1} H_n$. Thus we can rewrite the above exact sequence as

$$\Phi^{p-1} H_n \longrightarrow \Phi^p H_n \longrightarrow E^r_{p,q} \longrightarrow D^r_{p-1,q}.$$

Now

$$D^r_{p-1,q} = \operatorname{im}\left[(j^{p-2}j^{p-3}\cdots j^{p-r})_*\colon H_n(F^{p-r}) \to H_n(F^{p-1})\right].$$

But we can choose r big enough so that $p-r < s(n)$ to make $F_n^{p-r} = 0$. For such r, $D^r_{p-1,q} = 0$ and so by the first isomorphism theorem

$$E^r_{p,q} = \Phi^p H_n / \Phi^{p-1} H_n,$$

which proves the theorem. □

5. Spectral Sequences of a Total Complex

Now we specialize our discussion above to the first and second filtrations of the total complex of a special type of bicomplex.

Definition 5.1. A bicomplex (M, d', d'') is called a ***first quadrant bicomplex*** if $M_{p,q} = 0$ whenever $p < 0$ or $q < 0$.

Lemma 5.2. *If* $(\mathrm{Tot}(M), D)$ *is the total complex of a first quadrant complex* (M, d, d'')*, then the first and second filtrations of* $\mathrm{Tot}(M)$ *are bounded.*

Proof. From the definition of the first filtration of a total complex, we have ${}^{\mathrm{I}}F^n\mathrm{Tot}(M)_n = \mathrm{Tot}(M)_n$ and ${}^{\mathrm{I}}F^{-1}\mathrm{Tot}(M)_n = \{0\}$. Similarly, for the second filtration, we have ${}^{\mathrm{II}}F^n\mathrm{Tot}(M)_n = \mathrm{Tot}(M)_n$ and ${}^{\mathrm{II}}F^{-1}\mathrm{Tot}(M)_n = \{0\}$ (look at the corresponding lines in the lattice). □

Theorem 5.3. *Let* M *be a first quadrant bicomplex and let* ${}^{\mathrm{I}}E^r$ *and* ${}^{\mathrm{II}}E^r$ *be the spectral sequences arising from the first and second filtrations of* $\mathrm{Tot}(M)$*, respectively. Then*

$${}^{\mathrm{I}}E^2_{p,q} \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)) \quad \textit{and} \quad {}^{\mathrm{II}}E^2_{p,q} \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Proof. This is a direct consequence of **Theorem 4.9** and **Lemma 5.2**. □

Let (M, d', d'') be a bicomplex. Recall that each pth column $(M_{p,*}, d'')$ forms a complex, therefore we can take homology of pth columns to obtain a bigraded module, which we denote by $H''(M)$, whose (p,q) term is $H_q(M_{p,*})$. For each fixed q, the qth row $H''(M)_{*,q}$ of $H''(M)$

$$\ldots,\ H_q(M_{p-1,*}),\ H_q(M_{p,*}),\ H_q(M_{p+1,*}), \ldots$$

can be made into a complex if we define $\overline{d'}_{p,q}\colon H_q(M_{p,*}) \to H_q(M_{p-1,*})$ by

$$\overline{d'}_{p,q}([z]) = [d'_{p,q}(z)], \quad \text{for } [z] \in H_q(M_{p,*}).$$

(Check that $\overline{d'}_{p,q}$ is well-defined, using the fact that each square of the bicomplex anticommutes.) Now we can take homology of qth rows of $H''(M)$ to obtain a bigrade module, which we denote by $H'H''(M)$, whose (p,q) term is $H_p(H''(M)_{*,q})$. We call the bigraded module $H'H''(M)$ the ***first iterated homology*** of M and denote its (p,q) term by $H'_pH''_q(M)$.

Proposition 5.4. *If* (M, d', d'') *is a first quadrant bicomplex, then*

$${}^{\mathrm{I}}E^1_{p,q} = H_q(M_{p,*}) \quad \textit{and} \quad {}^{\mathrm{I}}E^2_{p,q} = H'_pH''_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Proof. Recall that $({}^{\mathrm{I}}E^r)$ arises from the first filtration $({}^{\mathrm{I}}F^p\mathrm{Tot}(M))$ of the total complex $(\mathrm{Tot}(M), D)$ of M. To simplify notation, we omit the prescript I for the rest of this proof and abbreviate ${}^{\mathrm{I}}F^p\mathrm{Tot}(M)_n$ to F^p_n. Recall also that (see **Proposition 3.9**)

$$E^1_{p,q} = H_n(F^p/F^{p-1}).$$

Now the map $\overline{D}_n\colon F^p_n/F^{p-1}_n \to F^p_{n-1}/F^{p-1}_{n-1}$ is induced from $D_n\colon F^p_n \to F^p_{n-1}$, where $D_n\colon \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$ given by

$$D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

By definition of the first filtration, $F_n^p/F_n^{p-1} = M_{p,q}$ and $F_{n-1}^p/F_{n-1}^{p-1} = M_{p,q-1}$ (here again we use the convention that $n = p+q$). To investigate the map $\overline{D}_n$, pick an element $a_n \in F_n^p/F_n^{p-1} = M_{p,q}$. Then think of a_n as an element in F_n^p, we have

$$\overline{D}_n a_n = D_n a_n = d'_{p,q} a_n + d''_{p,q} a_n \equiv d''_{p,q} a_n \mod F_{n-1}^{p-1}$$

since $d'_{p,q} a_n \in F_{n-1}^{p-1}$. Thus $\overline{D}_n = d''_{p,q}$ and so

$$E_{p,q}^1 = H_n(F^p/F^{p-1}) = \frac{\ker \overline{D}_n}{\operatorname{im} \overline{D}_{n+1}} = \frac{\ker d''_{p,q}}{\operatorname{im} d''_{p,q+1}} = H_q(M_{p,*}).$$

Now to compute $E_{p,q}^2$, recall (**Proposition 3.10**) that the differential $d^1 \colon E_{p,q}^1 \to E_{p-1,q}^1$ is given by

$$E_{p,q}^1 \xrightarrow{\gamma} D_{p-1,q} \xrightarrow{\beta} E_{p-1,q}^1,$$

where $E_{p,q}^1 = H_n(F^p/F^{p-1})$, $D_{p-1,q} = H_{n-1}(F^{p-1})$, $E_{p-1,q} = H_{n-1}(F^{p-1}/F^{p-2})$, γ is the connecting homomorphism and β is induced from the quotient map $F^{p-1} \to F^{p-1}/F^{p-2}$. Pick $[z] \in H_q(M_{p,*})$, where $z \in M_{p,q}$ such that $d''_{p,q} z = 0$. The connecting homomorphism arises from the following diagram,

$$\begin{tikzcd}
 & M_{p,q} \arrow[r] \arrow[d, "D_n"] & M_{p,q} \arrow[r] & 0 \\
0 \arrow[r] & M_{p-1,q} \arrow[r] & M_{p-1,q} &
\end{tikzcd}$$

which is easily seen to be just $d'_{p,q}$. Therefore the map $d^1 \colon E_{p,q}^1 = H_q(M_{p,*}) \to E_{p-1,q}^1 = H_q(M_{p-1,*})$ is given by

$$d^1([z]) = [d'_{p,q} z] \quad \text{for } [z] \in H_q(M_{p,*}).$$

Thus

$$E_{p,q}^2 = \frac{\ker \overline{d'}_{p,q}}{\operatorname{im} \overline{d'}_{p+1,q}} = H'_p H''_q(M),$$

as required. $\square$

Now we describe an analogous construction. Let (M, d', d'') be a bicomplex. Recall that each pth row $(M_{*,p}, d')$ forms a complex, therefore we can take homology of pth rows to obtain a bigraded module, which we denote by $H'(M)$, whose (p,q) term is $H_q(M_{*,p})$. For each fixed q, the qth column $H'(M)_{q,*}$ of $H'(M)$

$$\ldots,\ H_q(M_{*,p-1}),\ H_q(M_{*,p}),\ H_q(M_{*,p+1}), \ldots$$

can be made into a complex if we define $\overline{d''}_{p,q} \colon H_q(M_{*,p}) \to H_q(M_{*,p-1})$ by

$$\overline{d''}_{p,q}([z]) = [d''_{p,q}(z)], \quad \text{for } [z] \in H_q(M_{*,p}).$$

(Check that $\overline{d''}_{p,q}$ is well-defined, using the fact that each square of the bicomplex anticommutes.) Now we can take homology of qth columns of $H'(M)$ to obtain a bigrade module, which we denote by $H''H'(M)$, whose (p,q) term is $H_p(H'(M)_{q,*})$. We call the bigraded module $H''H'(M)$ the ***second iterated homology*** of M and denote its (p,q) term by $H''_p H'_q(M)$.

Proposition 5.5. *If (M, d', d'') is a first quadrant bicomplex, then*

$${}^{\mathrm{II}}E_{p,q}^1 = H_q(M_{*,p}) \quad \textit{and} \quad {}^{\mathrm{II}}E_{p,q}^2 = H''_p H'_q(M) \underset{p}{\Rightarrow} H_n(\operatorname{Tot}(M)).$$

Proof. Since the second filtration of $\mathrm{Tot}(M)$ is the first filtration of $\mathrm{Tot}(M^t)$, $({}^{\mathrm{II}}E^r)$ is the same as $({}^{\mathrm{I}}\widetilde{E}^r)$, the spectral sequence arising from the first filtration of $\mathrm{Tot}(M^t)$ (note that M^t is still a first quadrant bicomplex). From **Proposition 5.4**, we have

$$ {}^{\mathrm{II}}E^1_{p,q} = {}^{\mathrm{I}}\widetilde{E}^1_{p,q} = H_q(M^t_{p,*}) = H_q(M_{*,p}), $$

and

$$ {}^{\mathrm{II}}E^2_{p,q} = {}^{\mathrm{I}}\widetilde{E}^2_{p,q} = H_p(H''(M^t)_{*,q}) = H_p(H'(M)_{q,*}) = H''_pH'_q(M), $$

where we remind the readers that for the transpose bicomplex (M^t, δ', δ''), $\delta'_{p,q} = d''_{q,p}$ and $\delta''_{p,q} = d'_{q,p}$. $\square$

Definition 5.6. A spectral sequence (E^r, d^r) ***collapses*** on the p-axis if $E^2_{p,q} = \{0\}$ for all $q \neq 0$ (hence all the non-zero modules lie on the p-axis). A spectral sequence (E^r, d^r) ***collapses*** on the q-axis if $E^2_{p,q} = \{0\}$ for all $p \neq 0$ (hence all the non-zero modules lie on the q-axis).

Proposition 5.7. *Let (E^r, d^r) be a first quadrant spectral sequence (so it's either ${}^{\mathrm{I}}E^r$ or ${}^{\mathrm{II}}E^r$). Then*

(i) *if (E^r, d^r) collapses on either axis, then $E^\infty_{p,q} = E^2_{p,q}$ for all p, q;*
(ii) *if (E^r, d^r) collapses on the p-axis, then $H_n(\mathrm{Tot}(M)) \cong E^2_{n,0}$;*
if (E^r, d^r) collapses on the q-axis, then $H_n(\mathrm{Tot}(M)) \cong E^2_{0,n}$.

Proof. For part (i), suppose (E^r, d^r) collapses on the p axis. Consider a position (p, q). If $q \neq 0$, i.e. not on the p axis, then $E^2_{p,q} = 0$. It follows that $E^\infty_{p,q} = E^2_{p,q} = 0$. If $q = 0$, i.e. on the p axis, then notice that the map d^r has bidegree $(-r, r-1)$ and so it's never horizontal for $r > 2$. It follows that

$$ E^{r+1}_{p,0} = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,-r+1}} = E^r_{p,0} \quad \text{for } r \geq 2 $$

because the source of $d^r_{p+r,-r+1}$ and the target of $d^r_{p,q}$ are both 0. Thus $E^\infty_{p,0} = E^2_{p,0}$. A similar argument applies for the case where (E^r, d^r) collapses on the q-axis.

Now for part (ii), let's assume that (E^r, d^r) collapses on the q-axis. Since M is first quadrant, we have the following filtration of $H_n(\mathrm{Tot}(M))$:

$$ \{0\} = \Phi^{-1}H_n \subseteq \Phi^0 H_n \subseteq \cdots \subseteq \Phi^n H_n = H_n. $$

From **Theorem 5.3**, we know that $E^2_{p,q} \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M))$ (where $n = p + q$). In other words,

$$ \Phi^p H_n / \Phi^{p-1} H_n = E^\infty_{p,q} \quad \text{for } p + q = n. $$

From part (i), since (E^r, d^r) collapses on the q-axis, we have $E^\infty_{p,q} = 0$ for $p > 0$. It follows that

$$ \Phi^0 H_n = \Phi^1 H_n = \cdots = \Phi^n H_n = H_n. $$

When $p = 0$, we have $E^\infty_{0,q} = E^2_{0,p}$. Therefore,

$$ E^2_{0,p} = \Phi^0 H_n / \Phi^{-1} H_n = \Phi^0 H_n. $$

Thus $E^2_{0,p} = H_n(\mathrm{Tot}(M))$. The case where (E^r, d^r) collapses on the p axis can be proved similarly. $\square$

6. An Application

As an application of spectral sequences, we give a proof that Tor is independent of the variable resolved. We first recall the definition of Tor.

Definition 6.1. An R-module P is ***projective*** if for any R-modules G and H and homomorphisms $\alpha\colon P \to H$ and surjection $\beta\colon G \to H$, there exists a homomorphism ϕ making the diagram below commute.

$$\begin{tikzcd} & P \arrow[dl, "\phi"'] \arrow[d, "\alpha"] & \\ G \arrow[r, "\beta"'] & H \arrow[r] & 0 \end{tikzcd}$$

Lemma 6.2. *Every free R-module is projective.*

Proof. Suppose P is free. Let $B = \{b_i \colon i \in I\}$ be a basis for P. Since β is surjective, for each $b_i \in B$, there exists $g_i \in G$ such that $\beta(g_i) = \alpha(b_i)$. Define $\phi\colon B \to G$ by $\phi(b_i) = g_i$ (note that we have to invoke Axiom of Choice here) and extend by linearity. This gives a well-defined function $\phi\colon P \to G$ (see [1, **Proposition 2.34**]). Now to check that ϕ makes the diagram commute, pick $a = \sum_i r_i b_i \in P$. Then

$$\beta(a) = \beta\phi\left(\sum_i r_i b_i\right) = \beta\left(\sum_i r_i \phi(b_i)\right) = \sum_i r_i \beta(g_i) = \sum_i r_i \alpha(b_i) = \alpha(a),$$

as required. □

Definition 6.3. Let $\mathcal{X} = \{x_i \colon i \in I\}$ be a basis of a free R-module F and let $\mathcal{R} = \{\sum_i r_{ji} x_i \colon j \in J\}$ be a subset of F. If K is the submodule of F generated by $\mathcal{R}$, then we say that the module $M = F/K$ has ***generators*** $\mathcal{X}$ and relations $\mathcal{R}$ and the ordered pair $\langle \mathcal{X} | \mathcal{R} \rangle$ a ***presentation*** of M.

Lemma 6.4. *Every R-module M has a presentation.*

Proof. Let F be the free module generated by all the elements of M. Define $\phi\colon F \to M$ by $\phi(m) = m$ for $m \in M$ and extend by linearity. Clearly ϕ is surjective. By the first isomorphism theorem, we have $M = F/\ker\phi$, i.e. $M = \langle F | \ker\phi \rangle$. □

Definition 6.5. Let A be an R-module, a ***projective resolution*** of A is an exact sequence

$$P = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

in which each P_n is projective. A ***deleted projective resolution*** of A is the complex

$$P_A = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

Proposition 6.6. *Every R-module A has a projective resolution.*

Proof. We show that A has a free resolution, which in turn implies that A has a projective resolution since each free module is projective. We know that A has a presentation $\langle F_0 | K_1 \rangle$, which gives a short exact sequence:

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

Similarly, K_1 has a presentation $\langle F_1 | K_2 \rangle$, which gives a short exact sequence:

$$0 \longrightarrow K_2 \xrightarrow{i_2} F_1 \xrightarrow{\epsilon_1} K_1 \longrightarrow 0.$$

Now we splice these two sequences together to obtain

$$0 \longrightarrow K_2 \xrightarrow{i_2} F_1 \xrightarrow{i_1\epsilon_1} F_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

We need to check exactness at F_1 and F_0. Note that since ϵ_1 is surjective,

$$i_1\epsilon_1(F_1) = i_1(K_1) = \ker \epsilon.$$

Since i_1 is injective, $\ker i_1\epsilon_1 = \ker \epsilon_1 = \operatorname{im} \epsilon_2$. So the sequence is exact. This procedure can be continued indefinitely and we obtain a free resolution of A. $\square$

Before we proceed, we briefly recall the definition of a *left derived functor*. Let $\mathbf{Mod}_R$ denote the category of R-modules (the definitions work for more general abelian categories, but here we just focus on the category of R-modules for simplicity). Let $T\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ be a covariant, ***right exact*** functor, i.e. for any short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0,$$

the following sequence

$$T(L) \xrightarrow{T\psi} T(M) \xrightarrow{T\phi} T\phi T(N) \longrightarrow 0$$

is also exact (notice 0 lies on the right). Now for an R-module A, choose a deleted projective resolution of A:

$$P_A = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0.$$

Then apply the functor T to P_A to obtain the complex TP_A:

$$\cdots \longrightarrow T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \longrightarrow 0.$$

Now for each R-module A, the functors

$$(L_nT)A\colon = H_n(TP_A)$$

are called the ***left derived functors*** of T (the action of L_nT on morphisms is a bit more complicated and we do not need it in this notes). One important property of left derived functors is given in the following proposition.

Proposition 6.7. *Let A be an R-module and suppose $\widetilde{P}_A$ is another deleted projective resolution of A and let $\widetilde{L}_nT$ be the corresponding left derived functors. Then*

$$(L_nT)A \cong (\widetilde{L}_nT)A \quad \textit{for all } n \geq 0.$$

In other words, the definition of left derived functors does not depend on the choice of a deleted projective resolution of A.

Proof. For a proof, we refer the readers to [1, **Proposition 6.20**]. $\square$

Now we specialize our discussion to the functors $T = \square \otimes_R B$ and $S = A \otimes_R \square$.

Proposition 6.8. *The functors R and S are right exact.*

Proof. We refer the readers to [1, **Theorem 2.63**] for a proof. $\square$

Definition 6.9. If B is a left R-module and $T = \square \otimes_R B$, define

$$\operatorname{Tor}_n^R(\square, B) = L_nT.$$

In concrete terms, for each right R-module A, choose a deleted projective resolution of A:

$$P_A = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0.$$

Now form the complex $P_A \otimes_R B$:

$$\cdots \longrightarrow P_2 \otimes_R B \xrightarrow{d_2\otimes 1} P_1 \otimes_R B \xrightarrow{d_1\otimes 1} P_0 \otimes_R B \longrightarrow 0.$$

Then

$$\operatorname{Tor}_n^R(A,B) = H_n(P_A \otimes_R B).$$

Definition 6.10. If A is a right R-module and $S = A \otimes_R \square$, define

$$\operatorname{tor}_n^R(A,\square) = L_n S.$$

In concrete terms, for each left R-module B, choose a deleted projective resolution of B:

$$Q_B = \cdots \longrightarrow Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \longrightarrow 0.$$

Now form the complex $A \otimes_R Q_B$:

$$\cdots \longrightarrow A \otimes_R Q_2 \xrightarrow{1\otimes d_2} A \otimes_R Q_1 \xrightarrow{1\otimes d_1} A \otimes_R Q_0 \longrightarrow 0.$$

Then

$$\operatorname{tor}_n^R(A,B) = H_n(A \otimes_R Q_B).$$

Definition 6.11. A right R-module A is ***flat*** if $A \otimes_R \square$ is an exact functor; that is, whenever

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

is a short exact sequence of R-modules, the sequence

$$0 \longrightarrow A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M \xrightarrow{1\otimes\phi} A \otimes_R N \longrightarrow 0$$

is also exact. A left R-module B is ***flat*** if $\square \otimes_R B$ is an exact functor.

Proposition 6.12. *Every projective R-module is flat.*

Proof. We refer the readers to [1, **Proposition 3.46**] . □

Remark 6.13. Let A be a flat module, then it means in particular that if

$$L \xrightarrow{\psi} M \xrightarrow{\phi} N$$

is exact at M, then

$$A \otimes_R L \xrightarrow{1\otimes\psi} A \otimes_R M \xrightarrow{1\otimes\phi} A \otimes_R N$$

is exact at $A \otimes_R M$. This is because we have the short exact sequence

$$0 \longrightarrow \operatorname{im}\psi \longrightarrow M \longrightarrow \operatorname{im}\phi \longrightarrow 0.$$

Since A is flat we obtain the short exact sequence

$$0 \longrightarrow A \otimes_R \operatorname{im}\psi \longrightarrow A \otimes_R M \longrightarrow A \otimes_R \operatorname{im}\phi \longrightarrow 0.$$

Therefore

$$\frac{\ker(1\otimes\phi)}{\operatorname{im}(1\otimes\psi)} = \frac{\ker(1\otimes\phi)}{A \otimes_R \operatorname{im}\psi} = 0.$$

A similar argument works for the case $\square \otimes_R B$. ■

Given two complexes

$$P_A = \cdots \longrightarrow P_p \xrightarrow{\Delta'_p} P_{p-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

and

$$Q_B = \cdots \longrightarrow Q_q \xrightarrow{\Delta''_q} Q_{q-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow 0,$$

we can form a first quadrant bicomplex (M, d', d'') whose terms are given by

$$M_{p,q} = P_p \otimes_R Q_q, \quad d'_{p,q} = \Delta'_p \otimes 1, \quad d''_{p,q} = (-1)^p 1 \otimes \Delta''_q.$$

The total complex of M is called the ***tensor product of complexes*** and is denoted by $\operatorname{Tot}(M) = P_A \otimes_R Q_B$.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_0 \otimes Q_2 & \longleftarrow & P_1 \otimes Q_2 & \longleftarrow & P_2 \otimes Q_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_0 \otimes Q_1 & \longleftarrow & P_1 \otimes Q_1 & \longleftarrow & P_2 \otimes Q_1 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & P_0 \otimes Q_0 & \longleftarrow & P_1 \otimes Q_0 & \longleftarrow & P_2 \otimes Q_0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

To see that M is indeed a bicomplex, we just need to check that $d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0$. Plug in the definitions of d' and d'' we obtain

$$(\Delta'_p \otimes 1)(-1)^p(1 \otimes \Delta''_q) + (-1)^{p-1}(1 \otimes \Delta''_q)(\Delta'_p \otimes 1),$$

which is 0, as can easily be checked using the definition of tensor product.

Theorem 6.14. *Let P_A and Q_B be deleted projective resolutions of a right R-module A and a left R-module B, then* $\mathrm{Tor}_n^R(A,B) \cong \mathrm{tor}_n^R(A,B)$.

Proof. Let (M, d', d'') be the bicomplex as in the above paragraph whose total complex is $P_A \otimes Q_B$. The main idea of the proof is to look at ${}^{\mathrm{I}}E^r$ and ${}^{\mathrm{II}}E^r$. Let us first compute ${}^{\mathrm{I}}E^r$ using the first iterated homology. The pth column of M is

$$M_{p,*} = \longrightarrow P_p \otimes Q_{q+1} \longrightarrow P_p \otimes Q_q \longrightarrow P_p \otimes Q_{q-1} \longrightarrow .$$

Since P_p is projective, hence flat, this sequence is exact at every $q > 0$ (see **Remark 6.13**). Therefore $H_q(M_{p,*}) = 0$ for $q > 0$ and

$$H_0(M_{p,*}) = \mathrm{coker}(P_p \otimes_R Q_1 \to P_p \otimes_R Q_0).$$

Now $Q_1 \longrightarrow Q_0 \longrightarrow B \longrightarrow 0$ is exact, so by right exactness of tensor the sequence

$$P_p \otimes_R Q_1 \longrightarrow P_p \otimes_R Q_0 \longrightarrow P_p \otimes_R B \longrightarrow 0$$

is also exact. Therefore

$$H_0(M_{p,*}) = \mathrm{coker}(P_p \otimes_R Q_1 \to P_p \otimes_R Q_0) = P_p \otimes_R B$$

by the first isomorphism theorem. To sum up,

$${}^{\mathrm{I}}E^1_{p,q} = H_q(M_{p,*}) = \begin{cases} 0 & \text{if } q > 0, \\ P_p \otimes_R B & \text{if } q = 0. \end{cases}$$

Therefore,

$${}^{\mathrm{I}}E^2_{p,q} = H'_p H''_q(M) = \begin{cases} 0 & \text{if } q > 0, \\ H_p(P_A \otimes B) = \mathrm{Tor}_p^R(A,B) & \text{if } q = 0. \end{cases}$$

So the spectral sequence $({}^{\mathrm{I}}E^r)$ collapses on the p-axis. It follows from **Proposition 5.7** that $H_n(\mathrm{Tot}(M)) = {}^{\mathrm{I}}E^2_{n,0} = \mathrm{Tor}_n^R(A,B)$.

Now we compute ${}^{\mathrm{II}}E^r$ using the second iterated homology. The pth row of M is

$$M_{*,p} = \longrightarrow P_{q+1} \otimes Q_p \longrightarrow P_q \otimes Q_p \longrightarrow P_{q-1} \otimes Q_p \longrightarrow .$$

Since Q_p is projective, hence flat, we have $H_q(M_{*,p}) = 0$ for $q > 0$ and

$$H_0(M_{*,p}) = \mathrm{coker}(P_1 \otimes_R Q_p \to P_0 \otimes_R Q_p).$$

By the same argument as above we obtain

$$H_0(M_{*,p}) = A \otimes_R Q_B.$$

To sum up,

$$^{\mathrm{II}}E^1_{p,q} = H_q(M_{*,p}) = \begin{cases} 0 & \text{if } q > 0, \\ A \otimes_R Q_p & \text{if } q = 0. \end{cases}$$

Therefore,

$$^{\mathrm{II}}E^2_{p,q} = H''_p H'_q(M) = \begin{cases} 0 & \text{if } p > 0, \\ H_q(A \otimes Q_B) = \mathrm{tor}^R_q(A, B) & \text{if } p = 0. \end{cases}$$

So the spectral sequence $({}^{\mathrm{II}}E^r)$ collapses on the q-axis. It follows from **Proposition 5.7** that $H_n(\mathrm{Tot}(M)) = {}^{\mathrm{II}}E^2_{0,n} = \mathrm{tor}^R_n(A, B)$. Thus we conclude that

$$\mathrm{Tor}^R_n(A, B) = H_n(\mathrm{Tot}(M)) = \mathrm{tor}^R_n(A, B),$$

and so it does not matter which variable we resolve first. □

References

[1] Joseph J. Rotman. *An Introduction to Homological Algebra.* Springer, second edition, 2009.

[2] Joseph J. Rotman. *Advanced Modern Algebra.* American Mathematical Society, second edition, 2010.