MAT137 Lecture 18

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Agenda

- Rolle's theorem.
- The mean value theorem.

Recap

Theorem (Rolle's Theorem)

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

1. f *is continuous on* $[a, b]$,
2. f *is differentiable on* (a, b),
3. $f(a) = f(b)$,

THEN

$$\exists\ c \in (a, b) \ \text{ such that } \ f'(c) = 0.$$

Recap

Theorem (The Mean Value Theorem)

Let $a < b$. Let f be a function defined on $[a, b]$.
IF

1. *f is continuous on $[a, b]$,*
2. *f is differentiable on (a, b),*

THEN $\exists\ c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem for second derivative

Prove the following theorem

Theorem

Let f be a function which is twice differentiable on $[a, b]$ (at the endpoints we assume one-sided derivatives). IF

1. $f(a) = f'(a) = 0$, *and*
2. $f(b) = 0$,

THEN $\exists\ c \in (a, b)$ *such that* $f''(c) = 0$.

Hint. First apply Rolle's theorem to f and then to f'. Make sure to check all the hypotheses.

The Mean Value Theorem for second derivative

Prove the following theorem

Theorem

Let f be a function which is twice differentiable on $[a, b]$. THEN $\exists\, c \in (a, b)$ such that

$$\frac{f''(c)}{2} = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

Hint. Look at the following function

$$h(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)^2.$$

and apply Rolle's Theorem for second derivative.

Challenging Question

Can you formulate Rolle's Theorem and the Mean Value Theorem for n^{th} derivative?

The Cauchy Mean Value Theorem

Prove the following theorem

Theorem

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF

1. *f and g are continuous on $[a, b]$,*
2. *f and g are differentiable on (a, b),*

THEN $\exists\ c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Hint. Apply Rolle's Theorem to the following function

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x), \quad x \in [a, b].$$

Exercise. Show that the Mean Value Theorem is a special case of the Cauchy Mean Value Theorem.

How many zeros does a function have?

1. Show that the function $f(x) = x^{2017} - \cos x + 3x$ has exactly one zero.
2. Show that the function $g(x) = 3x^2 - \sin(2x) - 1$ has exactly two zeros.
3. Suppose h is twice differentiable on $\mathbb{R}$ and has three zeros. Show that f'' has at least one zero.

Zero-derivative implies constant

Theorem

Let $a < b$. Let f be a function defined on (a, b). IF $f'(x) = 0$ for all $x \in (a, b)$, then there exists a constant C such that $f(x) = C$ for all $x \in (a, b)$.

Question. Give an example of a continuous function that has zero derivative everywhere but is not constant.

Zero-derivative implies constant

Prove the following theorem

Theorem

Let $a < b$. Let f and g be functions defined on (a, b). IF

1. *f and g are differentiable on (a, b),*
2. *$f'(x) = g'(x)$ for all $x \in (a, b)$,*

THEN there exists a constant C such that

$$f(x) = g(x) + C, \quad \forall\, x \in (a, b).$$

Application of the MVT

Show that

$$|\tan b - \tan a| \geq |b - a|, \quad -\pi/2 < a, b < \pi/2.$$

$$|\arctan b - \arctan a| \leq |b - a|, \quad a, b \in \mathbb{R}.$$

Hint: Apply the MVT to $\tan x$ and $\arctan x$, respectively.

Next Class: Monday November 20

Watch videos 5.10, 5.11, 5.12 in Playlist 5.