MAT137 Lecture 20

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Agenda

- Sums and Σ notations.
- Suprema and infima.

Σ notation

Recall that

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Compute

(a) $\displaystyle\sum_{i=1}^{4}(2i-1)$

(b) $\displaystyle\sum_{i=1}^{4} 2i - 1$

Express the following sum using Σ notation

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots - \frac{x^{99}}{99!}.$$

Double Sums

Prove the following formulas by induction:

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}.$$

Use the above formulas to compute the following double sum

$$\sum_{i=1}^{N} \sum_{k=1}^{i} k.$$

Swapping summations

Decide what to write instead of each "?" so that the following identity is true

$$\sum_{i=1}^{N} \sum_{k=1}^{i} a_{i,k} = \sum_{k=?}^{?} \sum_{i=?}^{?} a_{i,k}.$$

Harmonic Sums

We define the N-**th harmonic term** as the sum

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} = \sum_{i=1}^{N} \frac{1}{i}.$$

Write the following sums in terms of harmonic terms

(a) $\displaystyle\sum_{i=1}^{N} \frac{1}{2i}.$

(b) $\displaystyle\sum_{i=1}^{N} \frac{1}{2i-1}.$

(c) $\displaystyle\sum_{i=1}^{2N} \frac{(-1)^{i-1}}{i}.$

Combining Sums

Fill in the "?" so that the following identity is correct, note that "?" should not contain x

$$\left[\sum_{k=1}^{N} x^k\right] + \left[\sum_{k=0}^{N} k x^{k+1}\right] = \left[\sum_{k=?}^{?} ? x^k\right] + ? x^{N+1}.$$

Sup, Inf, Max, Min

Definition

Let $A \subseteq \mathbb{R}$. We say that a is the **supremum** of A, denoted $a = \sup A$ if

- a is an **upper bound** of A, i.e. $\forall\, x \in A,\ x \leq a$,
- if b is an upper bound of A, then $a \leq b$.

We say that c is the **infimum** of A, denoted $c = \inf A$ if

- c is a **lower bound** of A, i.e. $\forall\, x \in A,\ x \geq c$,
- if d is an lower bound of A, then $d \leq c$.

Find the supremum, infimum, maximum, minimum of the following sets if they exist

(a) $A = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$.

(b) $B = \{2^n : n \in \mathbb{Z}\}$.

(c) $\emptyset$

The L.U.B. Axiom

The L.U.B. Axiom

Let $A \subseteq \mathbb{R}$. If

- A is bounded above, and
- A is not empty

Then A has a least upper bound, i.e. a supremum.

Assume the L.U.B. Axiom, prove the following

Theorem

Let $B \subseteq \mathbb{R}$. If

- B is bounded below, and
- B is not empty

Then B has a greatest lower bound, i.e. an infimum.

Equivalent Definition of Supremum

Definition

Let $A \subseteq \mathbb{R}$. We say that $a = \sup A$ if

- a is an upper bound of A
- $\forall\, \varepsilon > 0,\ \exists\, x \in A$ s.t. $x > a - \varepsilon$.

Write down the analogous definition for infimum.

Infima and suprema exercises

Let $A, B \subseteq \mathbb{R}$. Which of the following are true or false?
If false, find a counterexample.

(a) If $B \subseteq A$ and A is bounded above, then B is bounded above.

(b) If $B \subseteq A$ and B is bounded above, then A is bounded above.

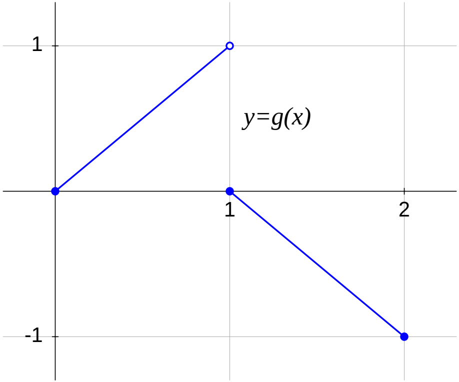
(c) If $B \subseteq A$ and A is bounded above, then $\sup B \leq \sup A$.

(d) If A and B are bounded above and $\sup A \leq \sup B$, then $A \subseteq B$.

(e) If A and B are bounded above,
then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

(f) If A and B are bounded above,
then $\sup(A \cap B) = \min\{\sup A, \sup B\}$.

Graph



Calculate:

(a) $\sup$ of g on $[0.5, 1.5]$

(b) $\max$ of g on $[0.5, 1.5]$

(c) $\inf$ of g on $[0.5, 1.5]$

(d) $\min$ of g on $[0.5, 1.5]$

Next Class: Monday January 8

Watch videos 7.5, 7.6, 7.7, 7.8, 7.9 in Playlist 7.