MAT137 Lecture 21

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January 8, 2018

Warm up: partitions

Which ones are partitions of $[0, 2]$?

(a) $[0, 2]$
(b) $(0, 2)$
(c) $\{0, 2\}$
(d) $\{1, 2\}$
(e) $\{0, 1.5, 1.6, 1.7, 1.8, 1.9, 2\}$

Partitions of different intervals

Let $a < b < c$.

(a) Let P be a partition of $[a, b]$.
Let Q be a partition of $[b, c]$.
How do we construct a partition of $[a, c]$ from them?

(b) Let R be a partition of $[a, c]$.
How do we construct partitions of $[a, b]$ and $[b, c]$ from it?

Warm-up: lower and upper sums

Let $f(x) = x^2$.
Consider the partition $P = \{0, 1, 2, 4\}$ of $[0, 4]$.
Calculate $U_P(f)$ and $L_P(f)$.

Lower and upper sums

Let f be a decreasing, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$

Which one (or ones) is a valid equation for $L_P(f)$? For $U_P(f)$?

(a) $\sum_{i=0}^{N} \Delta x_i \, f(x_i)$

(c) $\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$

(e) $\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$

(b) $\sum_{i=1}^{N} \Delta x_i \, f(x_i)$

(d) $\sum_{i=1}^{N} \Delta x_i \, f(x_{i+1})$

(f) $\sum_{i=0}^{N-1} \Delta x_{i+1} \, f(x_i)$

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

Definition of integral

Definition

Let f be a bounded function on $[a, b]$. We define the **lower integral of f from a to b** to be

$$\underline{I_a^b}(f) := \sup\{\text{lower sums of } f\}$$

and the **upper integral of f from a to b** to be

$$\overline{I_a^b}(f) := \inf\{\text{upper sums of } f\}.$$

We say that f is **integrable on** $[a, b]$ if $\underline{I_a^b}(f) = \overline{I_a^b}(f)$ and we denote the common value by

$$\int_a^b f(x)dx := \underline{I_a^b}(f) = \overline{I_a^b}(f).$$

An alternative definition

Exercise

The lower integral is the supremum of all the lower sums.
Try to write a definition of the lower integral that's similar to the alternative definition below.

Recall the equivalent definition of supremum we found last class:

Definition

If S is an upper bound of a set A, then S is the supremum of A when

$$\forall \epsilon > 0,\ \exists x \in A \text{ such that } S - \epsilon < x \leq S.$$

Example: a continuous function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Compute *all* the upper sums and *all* the lower sums.
3. Compute $\underline{I_0^4}(f)$
4. Compute $\overline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Next Class: Thursday January 11

Watch videos 7.5 to 7.12 in Playlist 7.