MAT137 Lecture 22

Huan Vo

University of Toronto

January 11, 2018

A non-continuous function

Consider the function f defined on $[0,1]$ given by

$$f(x) = \begin{cases} 0, & x = 0, \\ 5, & 0 < x \leq 1. \end{cases}$$

(a) Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0,1]$. What is $U_P(f)$?

(b) For the same partition, what is $L_P(f)$?

(c) What is the upper integral, $\overline{I_0^1}(f)$?

(d) What is the lower integral, $\underline{I_0^1}(f)$?

(e) Is f integrable on $[0,1]$?

A very non-continuous function

Consider the function defined on $[0,1]$:

$$f(x) = \begin{cases} 1 & \text{for } x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute *all* the lower sums.

(b) Find a partition P such that $U_P(f) = 1$.

(c) Find a partition P such that $U_P(f) < 1$.

(d) Find a partition P with only 4 points such that $U_P(f) = 0.52$.

(e) Find a partition P with only 4 points such that $U_P(f) = 0.5001$.

(f) For $\varepsilon > 0$, find a partition P such that $U_P(f) < \varepsilon$.

(g) Compute $\underline{I_0^1}(f)$ and $\overline{I_0^1}(f)$.

(h) Is f integrable on $[0,1]$?

A very, very non-continuous function

Consider the function defined on $[0,1]$:

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ 1 & \text{if } 1/2 < x \leq 1 \text{ and } x \in \mathbb{Q} \\ 0 & \text{if } 1/2 < x \leq 1 \text{ and } x \notin \mathbb{Q} \end{cases}$$

Question

Calculate $\overline{I_0^1}(f)$ and $\underline{I_0^1}(f)$. Is f integrable?

Hint: Think of different partitions and what their upper and lower sums are.

Computing an integral from Riemann sums

Exercise

Let $f(x) = x^2$ on $[0, 1]$.
Let $P_n = \{$breaking the interval into n equal pieces$\}$.

(a) Write a explicit formula for P_n.

(b) What is Δx_i?

(c) Write the Riemann sum $S^*_{P_n}(f)$ with sigma notation when we choose x_i^* as the right end-point.

(d) Add the sum

(e) Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.

(f) Repeat the last 3 questions when we choose x_i^* as the left end-point.

Helpful formulas: $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

More on upper/lower sums

- Consider $f(x) = x$ on $[0, 1]$.
  Is there a partition P of $[0, 1]$ such that $L_P(f) = U_P(f)$?

- Let f be a bounded function on $[a, b]$.
  Assume f is not constant.
  Prove that there exists a partition P of $[a, b]$ such that

  $$L_P(f) \neq U_P(f).$$

  Hint: This is easier than it looks.

Is this possible?

Find bounded functions f and g on $[0, 1]$ such that

- f is not integrable on $[0, 1]$,
- g is not integrable on $[0, 1]$,
- $f + g$ is integrable on $[0, 1]$.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \qquad \int_0^4 f(x)dx = 9, \qquad \int_0^4 g(x)dx = 2.$$

Compute:

(a) $\displaystyle\int_0^2 f(t)dt$

(b) $\displaystyle\int_0^2 f(t)dx$

(c) $\displaystyle\int_2^0 f(x)dx$

(d) $\displaystyle\int_2^4 f(x)dx$

(e) $\displaystyle\int_0^1 f(x)dx$

(f) $\displaystyle\int_0^4 [f(x) - 2g(x)]\ dx$

Next Class: Monday January 15

Watch videos 8.1, 8.2, 8.3 in Playlist 8.