MAT137 Lecture 24

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Agenda

FTC Part 1 and 2.

The Fundamental Theorem of Calculus Part 1

Theorem (FTC 1)

Let I be an interval. Let $a \in I$. Let f be a function on I. We define

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then F is continuous on I.
Moreover, if f is continuous, then F is differentiable and $F' = f$. In short, F is an anti-derivative of f.

An application of FTC 1

Exercise

Let f, u, v be differentiable functions with domain $\mathbb{R}$. Let us define

$$G(x) = \int_{u(x)}^{v(x)} f(t)\mathrm{d}t$$

Find a formula for

$$G'(x)$$

in terms of f, u, v, u', v'.

An application of FTC 1

Find the derivatives of the following functions

(a) $f(x) = \displaystyle\int_0^{\left(\int_0^{x^3} \frac{1}{1+\cos^2 t}\mathrm{d}t\right)} \frac{1}{1+\sin^2 t}\mathrm{d}t,$

(b) $g(x) = \displaystyle\int_1^{x^2} \left(\int_2^{y^3} \frac{1}{1+e^{t^2}+\sin^2 t}\mathrm{d}t\right)\mathrm{d}y.$

(c) $h(x) = \displaystyle\int_0^{x} x\xi(t)\mathrm{d}t$, where $\xi(t)$ is a continuous function.

True or False?

Let f and g be differentiable functions with domain $(-\infty, \infty)$. Assume that $f'(x) = g(x)$ for all x. Which of the following statements are always true?

(a) $f(x) = \int_0^x g(t) \mathrm{d}t$.

(b) If $f(0) = 0$, then $f(x) = \int_0^x g(t) \mathrm{d}t$.

(c) If $g(0) = 0$, then $f(x) = \int_0^x g(t) \mathrm{d}t$.

(d) There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t) \mathrm{d}t$.

(e) There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t) \mathrm{d}t$.

The Fundamental Theorem of Calculus Part 2

Theorem (FTC 2)

Let $a < b$*.*
Let f *be a continuous function on* $[a, b]$*.*
Let G *be any antiderivative* G *of* f*.*
Then

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a).$$

True, False, or Shrug?

We want to find a function H with domain $\mathbb{R}$ such that $H(1) = -2$ and such that $H'(x) = e^{\sin x}$ for all x. Decide whether each of the following statements is true, false, or we do not have enough information to decide.

(a) The function $H(x) = \int_0^x e^{\sin t} \mathrm{d}t$ is a solution.

(b) The function $H(x) = \int_1^x e^{\sin t} \mathrm{d}t$ is a solution.

(c) $\forall\ C \in \mathbb{R}$, the function $H(x) = \int_0^x e^{\sin t} \mathrm{d}t + C$ is a solution.

(d) $\exists\ C \in \mathbb{R}$ s.t. the function $H(x) = \int_0^x e^{\sin t} \mathrm{d}t + C$ is a solution.

(e) The function $H(x) = \int_1^x e^{\sin t} \mathrm{d}t - 2$ is a solution.

(f) There is more than one solution.

What is wrong?

We have

$$\int_{-1}^{1} \frac{1}{x^4} \mathrm{d}x = \left. \frac{-1}{3x^3} \right|_{-1}^{1} = -\frac{2}{3}.$$

However we know that $x^4 > 0$, so the above integral should be positive. What is the mistake here?

Definite Integrals

Evaluate the following definite integrals

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \csc x(\cot x - 3\csc x)\mathrm{d}x$

(b) $\displaystyle\int_{0}^{\pi/3} \left(\frac{2}{\pi}x - 2\sec^2 x\right)\mathrm{d}x$

(c) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\mathrm{d}x$

(d) $\displaystyle\int_{0}^{1} \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin^2 x}{1+\arctan^2 x + e^{x^2}}\right)\right]\mathrm{d}x$

(e) $\displaystyle\int_{-2}^{2} f(x)\mathrm{d}x$ where $f(x) = \begin{cases} 2, & -2 \leq x \leq 0, \\ 4 - x^2, & 0 < x \leq 2. \end{cases}$

An application of FTC 2

Let f be a function such that f' is continuous on $[a, b]$. Show that

$$\int_a^b f^2(x) f'(x) \mathrm{d}x = \frac{1}{3}[f^3(b) - f^3(a)].$$

Can you come up with a formula for

$$\int_a^b f^n(x) f'(x) \mathrm{d}x$$

for arbitrary integer $n \geq 0$?

More applications of FTC

Show that for all $x > 0$

$$\int_0^x \frac{1}{1+t^2}\mathrm{d}t + \int_0^{1/x} \frac{1}{1+t^2}\mathrm{d}t = \frac{\pi}{2}.$$

What happens when $x < 0$?

Next Class: Monday January 22

Watch videos 9.1, 9.2, 9.3, 9.4 in Playlist 9.