MAT137 Lecture 26

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January 25, 2018

Agenda

The substitution rule for definite integrals.

Integration by parts

The substitution rule for definite integrals

Let g' be continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\boxed{\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u.}$$

Remark. It is very important to change the limits of integration when we apply the substitution rule.

The substitution rule for definite integrals

The substitution rule for definite integrals

To apply the substitution rule for definite integrals we proceed as follows.

(i) Let

$$u = g(x),$$

then

$$\mathrm{d}u = g'(x)\mathrm{d}x,$$

(after this manipulation only the letter u should appear, not the letter x).

(ii) Change the limits of integration: $a \to g(a)$, $b \to g(b)$ (note that in general $g(a)$ can be $\geq g(b)$).

(iii) Find an antiderivative (as an expression involving u).

(iv) Apply FTC 2.

Substitution

Evaluate

$$\int_0^1 \frac{e^x + 1}{e^{2x} + 2xe^x + x^2 + 1}\mathrm{d}x.$$

Hint. Complete the square and let $u = e^x + x$.

Substitution

If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x) \mathrm{d}x = \frac{\pi}{2} \int_0^\pi f(\sin x) \mathrm{d}x.$$

Use the above formula to evaluate the integral

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \mathrm{d}x.$$

Substitution

Definition

A function f is called **periodic** if there exists a constant $P \neq 0$ such that

$$f(x+P) = f(x) \quad \text{for all } x.$$

If there exists a least positive constant P with this property, it is called the **period** of f.

Let f be a continuous periodic function with period $P > 0$, show that

$$\int_a^b f(x)\mathrm{d}x = \int_{a+P}^{b+P} f(x)\mathrm{d}x.$$

Integration by parts

If f' and g' are continuous, then

$$\boxed{\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x}$$

or in the differential notation

$$\boxed{\int u\mathrm{d}v = uv - \int v\mathrm{d}u}$$

For definite integrals, we obtain

$$\boxed{\int_a^b u\mathrm{d}v = uv\Big|_a^b - \int_a^b v\mathrm{d}u}$$

Integration by parts

Evaluate

$$\int x^3 e^{x^2} \mathrm{d}x.$$

Hint. Let $u = x^2$, $\mathrm{d}v = xe^{x^2}\mathrm{d}x$.

Integration by parts

Evaluate

$$\int \cos(\ln x)\mathrm{d}x.$$

Hint. Let $u = \cos(\ln x)$, $\mathrm{d}v = \mathrm{d}x$. Note that you need to apply integration by parts twice.

Integration by parts

Evaluate

$$\int_1^2 \frac{(\ln x)^2}{x^3} \mathrm{d}x.$$

Hint. Let $u = (\ln x)^2$, $\mathrm{d}v = \dfrac{\mathrm{d}x}{x^3}$. You need to apply integration by parts twice.

Substitution and integration by parts

Evaluate

$$\int e^{\sqrt[3]{x}}\mathrm{d}x.$$

Hint. First make the change of variable $t = \sqrt[3]{x}$ and then integration by parts.

Substitution and integration by parts

Evaluate

$$\int_0^{\pi} e^{\cos t} \sin(2t) \mathrm{d}t.$$

Hint. Recall that $\sin(2t) = 2 \sin t \cos t$. First make the change of variable $\heartsuit = \cos t$ and then integration by parts.

Reduction formula

Use integration by parts to show that

$$\int (\ln x)^n \mathrm{d}x = x(\ln x)^n - n \int (\ln x)^{n-1} \mathrm{d}x.$$

Use the above reduction formula to show that

$$\int_0^1 (\ln x)^n \mathrm{d}x = (-1)^n n!$$

for all integers $n \geq 0$.
Hint. Recall that

$$\lim_{x \to 0^+} x(\ln x)^n = 0$$

for $n \geq 0$ by L'Hopital's rule.

Next Class: Monday January 29

Watch videos 9.10, 9.11, 9.12, 9.13, 9.14 in Playlist 9.