MAT137 Lecture 28

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Agenda

Integration by trigonometric substitution

Partial fractions

Trigonometric substitution

Assume that $a \geq 0$.

(a) For $\sqrt{a^2 - x^2}$, use

$$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \text{then} \quad \sqrt{a^2 - x^2} = a\cos\theta.$$

(b) For $\sqrt{a^2 + x^2}$ use

$$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \text{then} \quad \sqrt{a^2 + x^2} = a\sec x.$$

(c) For $\sqrt{x^2 - a^2}$ use

$$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2} \quad \text{then} \quad \sqrt{x^2 - a^2} = a\tan\theta.$$

Trigonometric substitution

Evaluate the integral

$$\int_{\sqrt{2}/3}^{2/3} \frac{\mathrm{d}x}{x^5\sqrt{9x^2-1}}.$$

Trigonometric substitution

Evaluate the integral

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+5}}.$$

Partial fractions

(i) Consider a *rational function* $R(x) = P(x)/Q(x)$, i.e. $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

(ii) If $\deg P(x) \geq \deg Q(x)$ then we can perform long division to obtain

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)},$$

where $\deg r(x) < \deg Q(x)$.

(iii) Now we can write the rational function $r(x)/Q(x)$ as a sum of *partial fractions* as follows. The polynomial $Q(x)$ can be factored as a product of terms of the forms

$$(ax+b)^k, \quad \text{and} \quad (\alpha x^2 + \beta x + \gamma)^r,$$

where k, r are non-negative integers and $\beta^2 - 4\alpha\gamma < 0$, i.e. the quadratic polynomial has no real roots.

Partial fractions

Partial fractions

(a) Each factor $(ax+b)^k$ in the denominator gives rise to an expression of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}.$$

(b) Each factor $(\alpha x^2 + \beta x + \gamma)^r$ in the denominator gives rise to an expression of the form

$$\frac{A_1 x + B_1}{\alpha x^2 + \beta x + \gamma} + \frac{A_2 x + B_2}{(\alpha x^2 + \beta x + \gamma)^2} + \cdots + \frac{A_r x + B_r}{(\alpha x^2 + \beta x + \gamma)^r}.$$

Example

Express the following rational function in terms of partial fractions, you do not need to figure out the coefficients

$$\frac{2x^2+1}{(x-1)(x^2-3x+2)(x^2+4)^2}.$$

Integrals of rational functions

Evaluate the integral

$$\int \frac{x^3+1}{x^3+4x}\,\mathrm{d}x.$$

A very sneaky substitution

Let

$$t = \tan\frac{x}{2}, \quad \text{or} \quad x = 2\arctan t,$$

then

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t,$$

and

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

Use the above substitution to evaluate the integral

$$\int \frac{\mathrm{d}x}{3+5\sin x}.$$