MAT137 Lecture 29

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Agenda

Convergence of sequences

Bounded and monotonic sequence

Sequences

List the first five terms of the following sequences, where we assume a sequence starts at $n = 1$:

(a) $a_n = \dfrac{(-1)^n\sqrt{n}}{n! + 2}$

(b) $a_n =$ the nth prime number

(c) a_n is the number of edges of a regular n-gon

(d) $a_1 = 1, \quad a_{n+1} = \dfrac{a_n}{a_n + 2}$

(e) $a_1 = 1, \quad a_{n+1} = 1 + \dfrac{1}{1 + a_n}.$

The next term?

Guess a pattern for the following sequence and use it to come up with the next term.

$$\left\{0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right\}.$$

The limit of a sequence

Definition

We say that a sequence $\{a_n\}_{n=0}^{\infty}$ **converges** to the number $L \in \mathbb{R}$, denoted by

$$\lim_{n\to\infty} a_n = L$$

if for every $\varepsilon > 0$ there exists a natural number N such that whenever $n > N$ we have $|a_n - L| < \varepsilon$.

Exercise. From the definition of limit, show that

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2.$$

Subsequences

Definition

A **subsequence** of a sequence $\{a_n\}$ to be a sequence of the form

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots,$$

where the n_j are natural numbers with $n_1 < n_2 < n_3 < \ldots$. We can denote a subsequence as $\{a_{n_k}\}_{k=1}^{\infty}$.

For example, a subsequence of $\{a_n\}_{n=1}^{\infty}$ is

$$a_1, a_3, a_5, a_7, a_9, \ldots$$

Subsequences

Prove the following theorem

Theorem

Suppose that a sequence $\{a_n\}$ converges to L. Then any subsequence of $\{a_n\}$ also converges to L.

Exercise. Show that the sequences $(-1)^n$, $\sin(n)$ diverges.

As limit of a function

Prove the following theorem

Theorem

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* $n \geq 1$ *is an integer, then*

$$\lim_{n\to\infty} a_n = L.$$

Evaluate the following limits

(a) $\lim_{n\to\infty} \frac{1}{n^r}$, where $r > 0$.

(b) $\lim_{n\to\infty} \frac{\sin(\ln n)}{n}$.

(c) $\lim_{n\to\infty} \frac{\ln^2 n}{n}$.

Limit of absolute values

Prove the following theorem

Theorem

Let $\{a_n\}$ *be a sequence. Then*

$$\lim_{n\to\infty} |a_n| = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} a_n = 0.$$

Evaluate the following limit

(a) $\displaystyle \lim_{n\to\infty} \frac{(-1)^n}{n+2}$.

(b) $\displaystyle \lim_{n\to\infty} \frac{(-1)^n n^2}{e^n}$.

Composition of limits

Prove the following theorem

Theorem

If $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Again one cannot remove the continuity condition. Consider the sequence $a_n = 1/n$ and

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then

$$\lim_{n\to\infty} f(a_n) = 0 \neq f\left(\lim_{n\to\infty} a_n\right) = f(0) = 1.$$

Composition of limits

Evaluate the following limits

(a) $\lim_{n\to\infty} e^{-1/\sqrt[3]{n}}$.

(b) $\lim_{n\to\infty} \sin\left(\frac{\pi n+1}{2n}\right)$.

(c) $\lim_{n\to\infty} \arctan\left(\frac{n^2+1}{n^2+4}\right)$

(d) $\lim_{n\to\infty} \sqrt[n]{n}$.

Limit laws

Evaluate the following limits

(a) $\displaystyle\lim_{n\to\infty} \sqrt{\frac{1+3n^2}{2+n^2}}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{\arctan n}{n}$.

(c) $\displaystyle\lim_{n\to\infty} (n - \sqrt{n+1}\sqrt{n+2})$.

(d) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}\cos(n^{n^n})}{n+1}$.

Bounded and monotonic sequence

Definition

A sequence $\{a_n\}$ is called

(a) **increasing** if $a_n < a_{n+1} \; \forall \; n$,

(b) **decreasing** if $a_n > a_{n+1} \; \forall \; n$,

(c) **monotonic** if it is either increasing or decreasing,

(d) **bounded above** if there exists a number M such that $a_n \leq M \; \forall \; n$,

(e) **bounded below** if there exists a number m such that $a_n \geq M \; \forall \; n$,

(f) **bounded** if it is both bounded below and above.

Bounded and monotonic sequence

(a) Show that the following sequence

$$a_n = 2 - 3ne^{-n}, \quad n \geq 1$$

is increasing and bounded.

(b) Consider the following sequence

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}.$$

Show that $\{a_n\}$ is increasing and bounded above by 3.

Next Class: Monday February 12

Watch videos 11.6, 11.7, 11.8 in Playlist 11.