MAT137 Lecture 30

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February 12, 2018

Agenda

The monotone convergence theorem for sequences

The big theorem for sequences

The monotone convergence theorem for sequences

Theorem

IF a sequence is (eventually) monotonic and bounded,
THEN it is convergent.

More specifically,

Theorem

IF a sequence is (eventually) increasing and bounded above,
THEN it is convergent.

Theorem

IF a sequence is (eventually) decreasing and bounded below,
THEN it is convergent.

The monotone convergence theorem for sequences

Exercise. Suppose that $\{a_n\}_{n=1}^{\infty}$ is eventually increasing, show that $\{a_n\}$ is bounded below.

Exercise. Suppose that $\{b_n\}_{n=1}^{\infty}$ is eventually decreasing, show that $\{b_n\}$ is bounded above.

True or False?

Decide whether the following statements are true or false. If a statement is false, provide a counterexample.

(a) If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{n^2} = L$.

(b) If $\lim_{n\to\infty} a_{2n} = L$, then $\lim_{n\to\infty} a_n = L$.

(c) If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$.

(d) If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.

(e) If $\{a_n\}$ and $\{b_n\}$ are convergent and $b_n \neq 0$, then $\left\{\dfrac{a_n}{b_n}\right\}$ is convergent.

(f) If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.

(g) If $\{a_n\}$ converges to 0, then $\{(-1)^n a_n\}$ converges to 0.

Limit of a sequence

Let $\{a_n\}$ be the sequence given by

$$\begin{cases} a_1 = 1, \\ a_{n+1} = \sqrt{2 + a_n} \quad n \geq 1. \end{cases}$$

(a) Show that $\{a_n\}$ is increasing and bounded above by 3.

(b) Find $\lim\limits_{n \to \infty} a_n$.

The “Big Theorem” for sequences

Definition

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

$$a_n \ll b_n \text{ means } \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

We say that “$\{a_n\}$ is much smaller than $\{b_n\}$”.

Theorem

We have

$$\ln n \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0$, $c > 1$.

The "Big Theorem" for sequences

Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \dfrac{(-3)^n}{n!}$.

(b) $a_n = \dfrac{n!}{2^n}$.

(c) $a_n = \dfrac{2^{n^2}}{n!}$.

(d) $a_n = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)}{n!}$.

(e) $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$.

Next Class: Thursday February 15

Watch videos 12.1, 12.4, 12.7, 12.8 in Playlist 12.