MAT137 Lecture 31

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Agenda

Definitions of improper integrals.

The Basic Comparison Test (BCT).

Improper Integrals of "Type 1" (unbounded domain)

Definition

Let $a \in \mathbb{R}$.
Let f be a continuous function on $[a, \infty)$.
We define the integral of f from a to ∞ as

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\,\mathrm{d}x$$

provided this limit exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

Example. Evaluate the integral

$$\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x.$$

Improper Integrals of "Type 1" (unbounded domain)

Definition

Let $b \in \mathbb{R}$.
Let f be a continuous function on $(-\infty, b]$.
We define the integral of f from $-\infty$ to b as

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,\mathrm{d}x$$

provided this limit exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

Example. Evaluate the integral

$$\int_{-\infty}^{0} \frac{x}{x^4 + 4}\,\mathrm{d}x.$$

Improper Integrals of "Type 1" (unbounded domain)

Definition

Let $a \in \mathbb{R}$.
Let f be a continuous function on $(-\infty, \infty)$.
We define the integral of f from $-\infty$ to ∞ as

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{a}^{\infty} f(x)\,\mathrm{d}x.$$

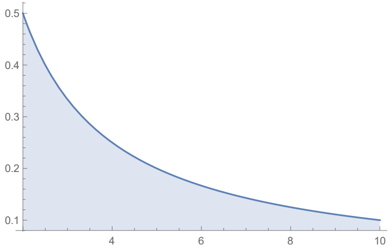
provided that each limit on the right hand side exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

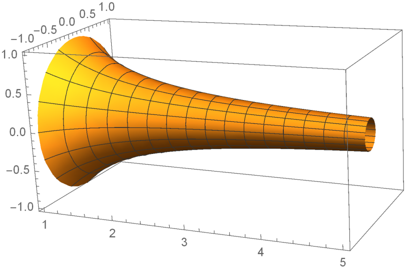
Example. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x, \quad \int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x.$$

Gabriel's Horn

Gabriel's Horn is the surface obtained by revolving the graph of $y = 1/x$, $1 \leq x < \infty$ about the x-axis





Gabriel's Horn

(a) Find the area of the region bounded by the x-axis and the graph of $y = 1/x$, $1 \leq x < \infty$.

(b) Find the volume of the Gabriel's Horn.

(c) The surface area of the surface obtained by revolving $y = f(x)$, $f(x) \geq 0$, $a \leq x \leq b$ about the x-axis is given by

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,\mathrm{d}x.$$

Show that the surface area of the Gabriel's Horn diverges to ∞.

So the Gabriel's Horn has finite volume but infinite surface area!

Improper Integrals of "Type 2" (bounded domain)

Definition

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\,\mathrm{d}x$$

provided this limit exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\,\mathrm{d}x$$

provided this limit exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

Improper Integrals of "Type 2" (bounded domain)

Show that

$$\boxed{\int_0^1 \frac{1}{x^p}\,\mathrm{d}x \text{ converges if and only if } p < 1.} \tag{1}$$

Contrast that with

$$\boxed{\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x \text{ converges if and only if } p > 1.} \tag{2}$$

Show that

$$\boxed{\int_0^1 x^p \ln x\,\mathrm{d}x \text{ converges if and only if } p > -1.} \tag{3}$$

Improper Integrals of "Type 2" (bounded domain)

Definition

If f has a discontinuity at c, where $a < c < b$, then we define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

provided each limit on the right hand side exists (as a finite number). In this case the integral is said to be **convergent**, otherwise it is said to be **divergent**.

Exercise. Evaluate the integral

$$\int_{-2}^1 \frac{\mathrm{d}x}{x^{2/5}}.$$

Improper Integrals of “Type 2” (bounded domain)

Recall that the following argument

$$\int_{-1}^{1} \frac{1}{x^4}\,\mathrm{d}x = -\frac{1}{3x^3}\bigg|_{-1}^{1} = -\frac{2}{3}$$

is WRONG because we cannot apply FTC 2 in this case. The integral is an improper integral. Show that it diverges.

Improper Integrals

Evaluate the improper integral

$$\int_0^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x$$

by writing it as

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x.$$

The first integral is improper of type 2 and the second integral is improper of type 1.

The Basic Comparison Test (BCT)

Most of the times we cannot evaluate improper integrals directly. But it is possible to check whether they converge or diverge using the following theorem.

Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.

(b) If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.

Exercise. The BCT still holds for improper integrals of type 2. Write down the statement for that case.

The BCT

Use the BCT to determine whether the following integral is convergent or divergent. Recall the basic integrals (1), (2), (3).

(a) $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,\mathrm{d}x$.

(b) $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt[3]{x}}\,\mathrm{d}x$.

(c) $\displaystyle\int_0^\infty \frac{\arctan(x^2)}{1+e^x}\,\mathrm{d}x$.

Next Class: Monday February 26

Watch videos 12.9, 12.10 in Playlist 12.