MAT137 Lecture 32

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Agenda

- The Basic Comparison Test (BCT).
- The Limit-Comparison Test (LCT).

The Basic Comparison Test (BCT)

Most of the times we cannot evaluate improper integrals directly. But it is possible to check whether they converge or diverge using the following theorem.

Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) *If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.*

(ii) *If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.*

Exercise. The BCT still holds for improper integrals of type 2. Write down the statement for that case.

Some important families of improper integrals

Recall that

$$\boxed{\int_0^1 \frac{1}{x^p}\,\mathrm{d}x \text{ converges if and only if } p < 1.} \tag{1}$$

Contrast that with

$$\boxed{\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x \text{ converges if and only if } p > 1.} \tag{2}$$

Show that

$$\boxed{\int_0^1 x^p \ln x\,\mathrm{d}x \text{ converges if and only if } p > -1.} \tag{3}$$

The Gaussian integral

Use the BCT to show that the integral

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x$$

is convergent. The integral is called the **Gaussian integral** (from the Gaussian distribution). In a multivariable calculus course we will learn that

$$\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

The Limit-Comparison Test ($L > 0$)

The Limit-Comparison Test ($L > 0$)

Let $a \in \mathbb{R}$.
Let f and g be *positive*, *continuous* functions on $[a, \infty)$.

- IF the limit
$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$$
exists (as a number), and $L > 0$.
- THEN
$$\int_a^\infty f(x)\,\mathrm{d}x \quad \text{and} \quad \int_a^\infty g(x)\,\mathrm{d}x$$
are both convergent or both divergent.

The same result also holds for improper integrals of type II (bounded domain).

The Limit-Comparison Test ($L = 0$)

The Limit-Comparison Test ($L = 0$)

Let $a \in \mathbb{R}$.

Let f and g be *positive*, *continuous* functions on $[a, \infty)$.

- IF
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$
- THEN
$$\int_a^\infty g(x)\,\mathrm{d}x \text{ converges} \implies \int_a^\infty f(x)\,\mathrm{d}x \text{ converges.}$$

The converse is NOT true: find two positive, continuous functions f and g on $[a, \infty)$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

and $\int_a^\infty f(x)\,\mathrm{d}x$ converges, but $\int_a^\infty g(x)\,\mathrm{d}x$ diverges.

The Limit-Comparison Test ($L = \infty$)

The Limit-Comparison Test ($L = \infty$)

Let $a \in \mathbb{R}$.

Let f and g be *positive*, *continuous* functions on $[a, \infty)$.

- IF
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.$$
- THEN
$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges } \implies \int_a^\infty g(x)\,\mathrm{d}x \text{ converges.}$$

The converse is NOT true: find two positive, continuous functions f and g on $[a, \infty)$ such that
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.$$
and $\int_a^\infty g(x)\,\mathrm{d}x$ converges, but $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.

Examples

Determine whether the following integrals are convergent or divergent

(a) $\displaystyle\int_1^\infty \frac{1-e^{-x^2}}{x^{2/3}}\,\mathrm{d}x.$

(b) $\displaystyle\int_1^\infty \frac{2-\cos^2(x)}{\sqrt{x}}\,\mathrm{d}x.$

(c) $\displaystyle\int_1^\infty \frac{x^2+\ln x}{\sqrt{3x^7+2}}\,\mathrm{d}x.$

(d) $\displaystyle\int_0^\pi \frac{(x^{2/3}+1)(3-\sin^2 x)}{x}\,\mathrm{d}x.$

(e) $\displaystyle\int_e^\infty \frac{1}{\sqrt{x+\sqrt{x}\ln x}}\,\mathrm{d}x.$

(f) $\displaystyle\int_1^\infty \left(1-\cos\left(\frac{1}{x^2}\right)\right)\mathrm{d}x.$

Hints.

(a) divergent.

(b) divergent.

(c) convergent.

(d) divergent.

(e) divergent.

(f) convergent.

Next Class: Thursday March 1

Watch videos (13.1), 13.2, 13.3, 13.4, 13.5, 13.6 in Playlist 13.