MAT137 Lecture 33

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Agenda

Series, what are they and why do we care?

Series

Series (or "infinite sums") appear quite frequently in mathematics. Some "cool" series are

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

The above series are instances of the **Riemann zeta function**:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1.$$

The Riemann hypothesis is one of the most famous open problems in mathematics, which is also a Millennium Prize Problem.

Series

Some more "cool" series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n}}{(2n)!} = -1 + \frac{\pi^2}{2!} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \cdots = 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n n} = \frac{1}{4} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots = \ln 2.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

$$\pi = 3.14159\cdots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \cdots$$

The definition of series

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth **partial sum**:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent** if the limit

$$\lim_{n\to\infty} s_n$$

exists as a finite number, say L. In this case L is called the **sum** of the series and we write $\sum_{n=1}^{\infty} a_n = L$. Otherwise $\sum_{n=1}^{\infty} a_n$ is said to be **divergent**.

Exercise. Give an ε-δ definition of what it means for a series to be convergent.

The definition of series

More concretely we can give an ε-δ definition of a convergent series as follows.

Definition

Given a series $\sum_{n=1}^{\infty} a_n$. Then

$$\sum_{n=1}^{\infty} a_n = L$$

means

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ s.t.} \,\forall\, n \in \mathbb{N},\ n > N \implies |s_n - L| < \varepsilon.$$

Exercise. Give an ε-δ definition of what it means for a series to be divergent.

The definition of series

Remark

One should distinguish between sequences and series. A series $\sum_{n=1}^{\infty} a_n$ is characterized by its sequence of partial sums $\{s_n\}_{n=1}^{\infty}$. On the other hand, given a sequence $\{a_n\}_{n=1}^{\infty}$ we can obtain a series $\sum_{n=1}^{\infty} a_n$ by adding the terms of the sequence. Note that the resulting series can converge or diverge. In particular $\lim a_n$ exists does NOT imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Example

Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2}.$$

(a) Write down the first few partial sums s_0, s_1, s_2, s_3 of the series.

(b) Find a general formula for the nth partial sum s_n.

(c) Compute the sum of the series.

Series are linear

Theorem

If the series $\sum a_n$ *and* $\sum b_n$ *are convergent, then so are the series* $c \sum a_n$ *(where* c *is a constant),* $\sum (a_n + b_n)$ *and* $\sum (a_n - b_n)$*, and*

(i) $\sum (ca_n) = c \sum a_n$

(ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$

(iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$

Exercise. Prove (i) from the definition of series.

True or False?

(a) If $\sum a_n$ is divergent then $\sum ca_n$ is divergent for any constant c.

(b) If $\sum a_n$ is divergent and $c \neq 0$ then $\sum ca_n$ is divergent.

(c) If $\sum a_n$ is convergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is divergent.

(d) If $\sum a_n$ and $\sum b_n$ are both divergent, then $\sum(a_n + b_n)$ is divergent.

(e) If the series $\sum a_n$ has positive terms and its partial sums are bounded above by 10 then $\sum a_n$ converges.

Hint. (a) F, (b) T, (c) T, (d) F, (e) T.

Geometric Series

Theorem

The geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is convergent if and only if $|r| < 1$ *and its sum is*

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

Geometric Series

Determine whether the following series is convergent or divergent. If it is convergent, find its sum.

(a) $-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$

(b) $\frac{3}{2}-\frac{9}{4}+\frac{27}{8}-\frac{81}{16}+\cdots$

(c) $\sum_{n=0}^{\infty}\frac{3^{n+2}}{(-4)^n}$

(d) $\sum_{n=1}^{\infty}\frac{2^n+(-1)^n4^n}{e^{2n}}$

(e) $\sum_{n=1}^{\infty}\left(\frac{2}{(n+1)(n+2)}+\frac{1}{3^n}\right)$

$0.9999\cdots = 1?$

Show that

$$0.\overline{9} = 0.99999\cdots = 1.$$

Hint. Write $0.99999\ldots$ as

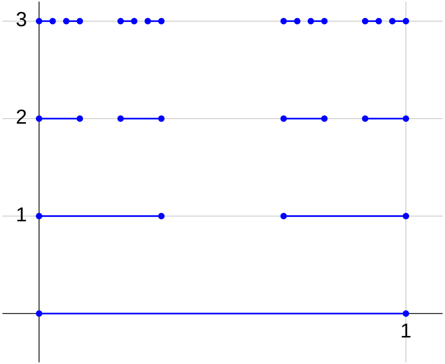
$$\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \sum_{n=1}^{\infty} \frac{9}{10^n}.$$

The Cantor Set (Optional)

The **Cantor Set** is a set that is defined as follows.

- Start with the interval $[0, 1]$.
- Remove the open middle third of $[0, 1]$, that leaves two closed intervals.
- Remove the open middle third of each of the two closed intervals, that leaves four closed intervals.
- Remove the open middle third of each of the the four closed intervals, that leaves eight open closed intervals .
- Keep removing the open middle third of each closed interval that remains from the previous step.
- The Cantor set consists of numbers that remain in $[0, 1]$ after all those intervals have been removed.

The Cantor Set (Optional)



Find the total length of all the intervals that are removed. Note that the Cantor set is not empty.

Next Class: Monday March 5

Watch videos 13.7, 13.8, 13.9 in Playlist 13.