MAT137 Lecture 35

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March 8, 2018

Agenda

- The integral test.
- The basic comparison test for series.
- The limit comparison test for series.
- The alternating series test.

The integral test

Theorem (The integral test)

Let $a \in \mathbb{R}$*.*
Let f *be a continuous, positive, decreasing function on* $[a, \infty)$*.*
Then

$$\int_a^{\infty} f(x)\,\mathrm{d}x \text{ is convergent} \iff \sum_n^{\infty} f(n) \text{ is convergent.}$$

Example (The p-series)

The p-series

$$\sum_n \frac{1}{n^p}$$

converges if and only if $p > 1$.

Remainder estimate for the integral test

Let f be a continuous, positive, decreasing function on $[1, \infty)$ and suppose that $f(k) = a_k$.

Suppose that $\sum_{k=1}^{\infty} a_k$ converges to s, and let

$$R_n = s - s_n = s - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

Draw a picture to illustrate the fact that

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x.$$

Applications of the integral test

Use the integral test to show that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

converges if and only if $p > 1$.

The basic comparison test for series

Theorem (BCT for series)

Let $\sum_{n}^{\infty} a_n$ *and* $\sum_{n}^{\infty} b_n$ *be two series.*
Suppose that

$$0 \leq a_n \leq b_n \text{ for all } n > N,$$

where N *is some positive integer. Then*

(i) $\sum_{n}^{\infty} b_n$ *converges* $\Longrightarrow \sum_{n}^{\infty} a_n$ *converges.*

(ii) $\sum_{n}^{\infty} a_n$ *diverges* $\Longrightarrow \sum_{n}^{\infty} b_n$ *diverges.*

The BCT for series

Determine whether the following series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n^2}}{n^{0.7}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Hint. (a) diverges, (b) converges, (c) converges.

The limit comparison test for series

Theorem (LCT for series)

Let $\sum_n^\infty a_n$ *and* $\sum_n^\infty b_n$ *be two positive series.*
If the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists (as a finite number) and $L > 0$.
Then

$$\sum_n^\infty a_n \textit{ is convergent } \iff \sum_n^\infty b_n \textit{ is convergent.}$$

Exercise. Show that

- If $L = 0$, then $\sum_n^\infty b_n$ converges $\Longrightarrow \sum_n^\infty a_n$ converges.
- If $L = \infty$, then $\sum_n^\infty a_n$ converges $\Longrightarrow \sum_n^\infty b_n$ converges.

The LCT for series

Determine whether the following series converges or diverges

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{\sqrt{n^5+\ln n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Hint. (a) diverges, (b) diverges, (c) converges, (d) diverges.

Alternating series test

Theorem (Alternating Series Test)

Consider a series of the form

$$\sum_{n}^{\infty}(-1)^n b_n \quad \textit{or} \quad \sum_{n}^{\infty}(-1)^{n+1} b_n.$$

IF

(i) $b_n > 0$ *for all* n.

(ii) b_n *is decreasing.*

(iii) $\lim_{n\to\infty} b_n = 0$.

Then the series is convergent.

Remainder estimation for alternating series

Theorem (Alternating Series Estimation Theorem)

For an alternating series that satisfies the conditions of the alternating series test, let R_n denote the remainder

$$R_n = s - s_n,$$

where s is the sum of the series and s_n is the nth partial sum. Then we have

$$|R_n| \leq |b_{n+1}|.$$

Alternating series test example

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-0.2)^n}{n!}.$$

(a) Show that the series is convergent.

(b) How many terms of the series do we need to add so that the error is less than 0.001.

True or False?

(a) If $\sum a_n^2$ converges, then $\sum a_n$ converges.

(b) If $\sum a_n$ is a convergent series with positive terms, then $\sum a_n^2$ converges.

(c) If $\sum a_n$ and $\sum b_n$ diverge, then $\sum a_n b_n$ diverges.

(d) If $\sum a_n$ is a convergent series with positive terms, then $\sum \sin(a_n)$ is convergent.

(e) If $\sum a_n$ and $\sum b_n$ converge, then $\sum a_n b_n$ converges.

(f) If $\sum a_n$ is a positive series and $\lim_{n\to\infty} \sqrt{n} a_n = L \neq 0$, then $\sum a_n$ converges.

Hint. (a) false, (b) true, (c) false, (d) true, (e) false, (f) false.

Next Class: Monday March 12

Watch videos 13.5, 13.6, 13.7 in Playlist 13.