# MAT137 Lecture 37

Huan Vo

University of Toronto

March 15, 2018

# Agenda

- The ratio test.
- Power series.

# The ratio test

## Theorem (The ratio test)

*Let* $\sum_{n}^{\infty} a_n$ *be a series. Assume* $a_n \neq 0$ *for all* $n$*. Assume the limit*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \textit{exists or is } \infty.$$

- *IF* $L < 1$, *THEN* $\sum_{n}^{\infty} a_n$ *converges absolutely.*
- *IF* $L = 1$, *THEN no conclusion can be drawn.*
- *IF* $L > 1$ *or* $L = \infty$, *THEN* $\sum_{n}^{\infty} a_n$ *diverges.*

**Exercise.** In the case that $L = 1$, give an example of a series that converges and an example of a series that diverges.

# Convergence or divergence?

Determine whether the series is convergent or divergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

(d) $\displaystyle\frac{2}{3} + \frac{2 \cdot 5}{3 \cdot 5} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!}$

# The root test

## Theorem

*Let* $\sum_{n}^{\infty} a_n$ *be a series. Suppose that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1.$$

*Show that* $\sum_{n}^{\infty} a_n$ *converges absolutely.*

**Hint.** Let $L < r < 1$. Show that for $n$ big enough we have $|a_n| < r^n$.

# The root test

## Theorem (The root test)

*Let* $\sum_{n}^{\infty} a_n$ *be a series. Assume* $a_n \neq 0$ *for all* $n$*. Assume the limit*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} \quad \textit{exists or is } \infty.$$

- *IF* $L < 1$, *THEN* $\sum_{n}^{\infty} a_n$ *converges absolutely.*
- *IF* $L = 1$, *THEN no conclusion can be drawn.*
- *IF* $L > 1$ *or* $L = \infty$, *THEN* $\sum_{n}^{\infty} a_n$ *diverges.*

# Root test vs ratio test

Consider the following series $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \begin{cases} n/3^n, & n \text{ odd}, \\ 1/3^n, & n \text{ even} \end{cases}$$

(a) Show that the ratio test does not apply.

(b) Use the root test to show that $\sum a_n$ converges.

# Power series

## Definition

Let $a \in \mathbb{R}$.

A **power series centered at** $a$ is a function $f$ defined by an equation like

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where $c_0, c_1, c_2, \ldots \in \mathbb{R}$.

# Power series

More specifically, the function $f$ is defined as follows.

- For $x_0 \in \mathbb{R}$, we have a series

$$c_0 + c_1(x_0 - a) + c_2(x_0 - a)^2 + \cdots$$

- If the series converges to $s$, then we define

$$f(x_0) = s.$$

  Otherwise we say that $f(x_0)$ is undefined, or $x_0$ is not in the domain of $f$.
- Note that $a$ is always in the domain of $f$, why?

# The geometric series

Let $f$ be the function defined by

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

(a) Evaluate $f(0)$, $f(1/2)$, $f(-1/2)$.

(b) Find the domain of $f$.

# The Bessel function

The **Bessel function of order 0** is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Use the ratio test to find the domain of $J_0$.

# Next Class: Monday March 19

Watch videos 14.3, 14.4 in Playlist 14.