MAT137 Lecture 38

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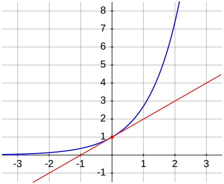
Agenda

Definitions of Taylor polynomials

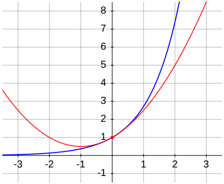
Taylor polynomials

GOAL: approximate functions by polynomials.

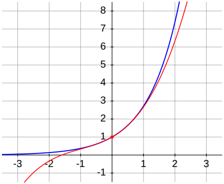
In the following figures the function $f(x) = e^x$ (blue curve) is approximated by polynomials of higher and higher degrees.



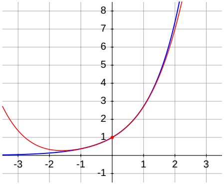
Degree 1



Degree 2



Degree 3



Degree 4

C^n functions

Definition

A function f is called

- C^0 when f is continuous.
- C^1 when f' exists and is continuous.
- C^2 when f' and f'' exist and are continuous
- $\ldots$
- C^n when f', f'', ..., $f^{(n)}$ exist and are continuous.
- C^∞ when all derivatives exist (and are continuous).

Example. The function

$$f(x) = \begin{cases} x^2/2, & x \geq 0, \\ -x^2/2, & x < 0. \end{cases}$$

is C^1 but not C^2.

Taylor polynomials

Definition (Taylor polynomials)

Let $a \in \mathbb{R}$.
Let f be a C^n function near a.
The n**th Taylor polynomial of** f **centered at** a is the polynomial $P_n(x)$ of degree $\leq n$ that satisfies

$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0.$$

Exercise. Show that $P_1(x) = f(a) + f'(a)(x-a)$, which is precisely the equation of the tangent line to f at a.

Taylor polynomials

Theorem

Let $a \in \mathbb{R}$.
Let f be a C^n function near a.
THEN the nth Taylor polynomial of f centered at a is the polynomial $P_n(x)$ given by

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

We will prove this theorem.

Taylor polynomials

Hint.

- First write
$$P_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n.$$
We need to find the coefficients c_0, c_1, …, c_n.
- Show that
$$P_n(a) = f(a),\ P_n'(a) = f'(a),\ \ldots,\ P_n^{(n)}(a) = f^{(n)}(a).$$
(**Hint.** Use L'Hôpital's rule repeatedly.)
- Show that
$$P_n^{(k)}(a) = k!c_k, \quad 0 \le k \le n.$$
Conclude that
$$c_k = \frac{f^{(k)}(a)}{k!}, \quad 0 \le k \le n,$$
as required.

Taylor polynomials

Exercise. Find the 4th Taylor polynomial of $f(x) = e^{2x}$ centered at 1.

Next Class: Thursday March 22

Watch videos 14.5, 14.6, 14.7, 14.8 in Playlist 14.