MAT137 Lecture 39

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Agenda

- Some Taylor series.
- Analytic functions and the remainder theorems.

Taylor series

Definition (Taylor series)

Let $a \in \mathbb{R}$.
Let f be a C^∞ function at a.
The **Taylor series of f at a** is the power series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

When $a = 0$ we call the Taylor series the **Maclaurin series**.

Warning. In general $S(x) \neq f(x)$.

Analytic functions

Definition (Analytic functions)

Let f be a C^{∞} function defined on an open interval I.
Let $a \in I$. Let $S(x)$ be the Taylor series of f at a.
We say that f **is analytic at** a when there exists an interval J_a centered at a such that

$$f(x) = S(x)$$

for all $x \in J_a$.
We say that f **is analytic on** I if f is analytic at a for all $a \in I$.

Taylor series

Exercise. Let f be a polynomial of degree n

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

Let $a \in \mathbb{R}$. Show that the Taylor series of f at a is f itself. Conclude that f is analytic on $\mathbb{R}$.

Taylor series

Exercise. Let f be defined by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{k=0}^{\infty} c_k(x-a)^k$$

with radius of convergence $R > 0$. Show that the Taylor series of f at a is f itself.

Taylor series

Exercise. Find the Taylor series of $f(x) = e^{3x}$ centered at 2 and state the radius of convergence.

Lagrange's Remainder Theorem

Theorem

Let I be an open interval. Let $a \in I$.
Let f be a C^{n+1} function on I.
Let P_n be the nth Taylor polynomial of f at a.
We call $R_n(x) = f(x) - P_n(x)$ the **remainder***.*
THEN there exists ξ between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Exercise. Show that f is analytic at a if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

Lagrange's Remainder Theorem

Exercise. Consider the function $f(x) = \sin(2x)$.

(a) Find the Maclaurin series $S(x)$ of f and state it radius of convergence.

(b) Use the Lagrange's remainder theorem to show that

$$f(x) = S(x)$$

for all x in the interval of convergence of $S(x)$.

Some common Maclaurin series

Some common Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad R = 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad R = \infty.$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty.$$

Exercise. Find the Taylor series of $\cos x$, $\ln(1+x)$ and their radii of convergence.

The binomial series

Consider the function $f(x) = (1+x)^k$, $k \in \mathbb{R}$. Show that the Maclaurin series of f is given by

$$S(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots ,$$

where

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

are called **binomial coefficients**, which is 1 when $n = 0$. It can be shown that

$$f(x) = S(x) \quad \text{for } |x| < 1.$$

Exercise. Show that when k is a non-negative integer we recover the **binomial theorem**.

More Taylor series

Consider the function

$$f(x) = \frac{1}{1+x^2}.$$

(a) Find the Maclaurin series of f by using a common Maclaurin series and state its radius of convergence.

(b) Find the Maclaurin series of $\arctan x$ and state its radius of convergence.

(c) Find the sum of the series

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots$$

Taylor polynomials

Show that if P is an nth-degree polynomial, then

$$P(x+1) = \sum_{i=0}^{n} \frac{P^{(i)}(x)}{i!}.$$

Next Class: Monday March 26

Watch videos 14.9, 14.10 in Playlist 14.