MAT137 Lecture 4

Huan Vo

University of Toronto

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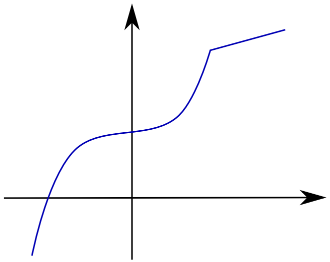
Agenda

- Definitions and Proofs.
- Induction proofs.

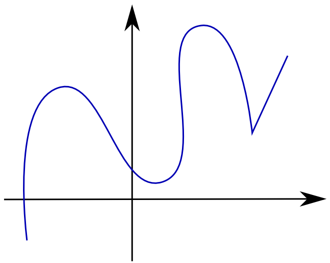
Injectivity

Definition

A function f defined on a domain D is called **injective** on D (or **one-to-one** on D) if different inputs to the function always yield different outputs.



one to one



not one to one

Injectivity

Choose the correct definition of one-to-one functions.

Let f be a function with domain $D \subseteq \mathbb{R}$. We say that f is **one-to-one** if

1. $f(x_1) \neq f(x_2)$.
2. $\exists\, x_1, x_2 \in D \quad \text{s.t.} \quad f(x_1) \neq f(x_2)$.
3. $\forall\, x_1, x_2 \in D, \quad f(x_1) \neq f(x_2)$.
4. $\forall\, x_1, x_2 \in D, \quad x_1 \neq x_2, \quad f(x_1) \neq f(x_2)$.
5. $\forall\, x_1, x_2 \in D, \quad x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$.
6. $\forall\, x_1, x_2 \in D\ f(x_1) \neq f(x_2) \ \Rightarrow\ x_1 \neq x_2$.
7. $\forall\, x_1, x_2 \in D\ f(x_1) = f(x_2) \ \Rightarrow\ x_1 = x_2$.

Injectivity

Show that the function $f(x) = x^2$, $x \in [0, 1]$ is one-to-one.

Proof.

Using definition **7**, take $x_1, x_2 \in [0, 1]$ and suppose that $f(x_1) = f(x_2)$, i.e.

$$x_1^2 = x_2^2.$$

Since $x_1, x_2 \geq 0$, taking square root of both sides we obtain $x_1 = x_2$. So f is one-to-one, as required. □

Injectivity

Show that the function $f(x) = x^2$, $x \in [-1, 1]$ is not one-to-one.

Proof.

To show that f is not one-to-one, we negate definition **7**, i.e.

$$\exists\ x_1, x_2 \in D \text{ s.t. } f(x_1) = f(x_2), \text{ but } x_1 \neq x_2.$$

We can simply choose $x_1 = -1$, $x_2 = 1$, then $f(-1) = f(1) = 1$, but $x_1 \neq x_2$. $\square$

Induction

To prove a statement S_n is true for all $n \geq 1$, we can proceed as follows.

1. **Base Case:** Prove that S_1 (or some other starting point) is true.
2. **Induction Hypothesis:** Prove that $\forall n \geq 1$,

$$S_n \text{ is true} \implies S_{n+1} \text{ is true.}$$

Induction

Suppose we have some statements S_n for all $n \geq 1$.

In each of the following cases, which S_n's will we know are true?

1. **Case 1:** Suppose we have shown that:
   - S_7 is true.
   - $\forall\ n \geq 1$, S_n is true $\implies S_{n+1}$ is true.
2. **Case 2:** Suppose we have shown that:
   - S_1 is true.
   - $\forall n \geq 7$, S_n is true $\implies S_{n+1}$ is true.
3. **Case 3:** Suppose we have shown that:
   - S_1 is true.
   - $\forall n \geq 1$, S_{n+1} is true $\implies S_n$ is true.
4. **Case 4:** Suppose we have shown that:
   - S_1 is true.
   - $\forall n \geq 1$, S_n is true $\implies S_{n+3}$ is true.

Induction

Figure out what goes wrong in the following induction proof.

Theorem

All aliens have the same color??

Proof?

We will prove this by induction on the number of aliens.

- **Base Case:** S_1 is true, since with just one alien, all aliens have the same color.
- **Induction Step:** Assume S_n, which is the statement that all n aliens have the same color. Now given a set of $n+1$ aliens $\{a_1, a_2, \ldots, a_{n+1}\}$, we can conclude by the induction hypothesis that $\{a_1, a_2, \ldots, a_n\}$ all have the same color, and $\{a_2, a_3, \ldots, a_{n+1}\}$ all have the same color. Since $\{a_2, \ldots, a_n\}$ belongs to both sets, it follows that $\{a_1, \ldots, a_{n+1}\}$ have the same color, as required.

$\square$

Next Class: Thursday Sept 21

Watch videos 1, 2, 3, 4 in Playlist 2.