MAT137 Lecture 41

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Agenda

Applications of Taylor series.

Estimation

Approximate

$$\int_0^1 \sin(x^2)dx$$

to within four decimal places.

Hint.

- Write the Maclaurin series of $\sin(x^2)$.
- Integrate term by term.
- What is the resulting series? Can you estimate the remainder?

Estimating π

- Find the Maclaurin series of $\arctan x$. What is the radius of convergence?
- It can be shown that the Maclaurin series converges to $\arctan x$ when $x = 1$. Prove that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

- How many terms do we need to add to approximate π up to ten decimal places?

Estimation

Use the Maclaurin series of $\ln(x+1)$ to estimate $\ln 1.4$ to within 0.001.

Lagrange's Remainder Theorem

Theorem

Let I be an open interval. Let $a \in I$.
Let f be a C^{n+1} function on I.
Let P_n be the nth Taylor polynomial of f at a.
We call $R_n(x) = f(x) - P_n(x)$ the **remainder***.*
THEN there exists ξ between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Maximum error

We want to approximate e^x by its second degree Taylor polynomial

$$e^x \approx 1 + x + \frac{x^2}{2!}.$$

Can you find $a > 0$ such that for all $x \in (-a, a)$ the above approximation is accurate within three decimal places?

Taylor series and limits

Evaluate the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{e^{-x^2} - 1 + x^2}{x(\arctan x)^3}$.

(b) $\displaystyle\lim_{x\to 0} \frac{3\cos(x^2) - 1}{x^3 \sin(2x)}$.

(c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x\sin(2x)}$.

Hint. Recall that

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \text{h.o.t.}, \quad |x| < 1.$$

Taylor series and integrals

Evaluate the following integrals

(a) $\displaystyle\int_0^1 \frac{\sin x}{x} dx.$

(b) $\displaystyle\int_0^{0.5} \frac{1-\cos x}{x^2} dx.$

(c) $\displaystyle\int_0^1 x^2 e^{-x^2} dx.$

(d) $\displaystyle\int_0^1 \cos(\sqrt{x}) dx.$

Taylor series and sums

Evaluate the following sum

$$\sum_{n=0}^{\infty} \frac{(n+1)(-1)^n x^{2n+1}}{(2n+1)!}.$$

Hint. Start with the Maclaurin series of $\sin x$.

Taylor series and sums

Evaluate the following sum

$$\sum_{n=0}^{\infty} \frac{1}{(4n+1)9^n}.$$

Hint. Start with the geometric series.

Taylor series and sums

Find the Maclaurin series of

$$f(x) = \frac{e^x - 1}{x}.$$

Use it to evaluate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}.$$

Next Class: Monday April 2

Watch videos 14.15 in Playlist 14.