MAT137 Lecture 7

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Agenda

- How to show that a limit DNE.
- Limit laws.

Limit does not exist

Write the formal definition of

$$\lim_{x \to a} f(x) \neq L.$$

Hint: Negate the definition of $\lim\limits_{x \to a} f(x) = L$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Limit does not exist

Write the formal definition of

$$\lim_{x \to a} f(x) \text{ D.N.E.}$$

Hint: Negate the statement

$$\exists L, \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Limit does not exist

Using the ε-δ definition, show that

$$\lim_{x \to 0} \frac{x}{|x|} \text{ D.N.E.}$$

Hint:

1. Plot the function $x/|x|$ near 0.
2. For any $L \in \mathbb{R}$, what $\varepsilon > 0$ should you choose?
3. Construct a formal proof.

Side Limits

Prove the following theorem

Theorem

Let $a \in \mathbb{R}$. Let f be a function defined, at least, on an interval centered at a, except possibly at a. Let $L \in \mathbb{R}$. Show that $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ if and only if $\lim_{x \to a} f(x) = L$.

Hint: For the *only if* direction

1. Let $\varepsilon > 0$ be given.
2. Write down the formal definition of $\lim_{x \to a^+} f(x) = L$.
3. Write down the formal definition of $\lim_{x \to a^-} f(x) = L$.
4. What $\delta > 0$ should you choose?

Side Limits

Use the above theorem to show that

$$\lim_{x \to 0} \frac{x}{|x|} \text{ D.N.E.}$$

A bad proof

Bad Theorem

Let f and g be functions defined near a, except possibly at a. If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$.

Bad Proof

We have

$$\lim_{x \to a} f(x)g(x) = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right] = 0 \cdot \left[\lim_{x \to a} g(x)\right] = 0,$$

because 0 times anything is 0.

- Find the error in the proof.
- Show the theorem is false with a counterexample.

Limits of products

Find functions f and g such that

$$\lim_{x \to 0} f(x) = 0, \text{ and } \lim_{x \to 0} [f(x)g(x)] = 7.$$

Limits of products

Find functions f and g such that

$$\lim_{x\to 0} f(x) = 0, \text{ and } \lim_{x\to 0}[f(x)g(x)] = \infty.$$

Limits of sums

Find functions f and g such that

$$\lim_{x\to 0} f(x) \text{ D.N.E., and } \lim_{x\to 0}[f(x)+g(x)] = 7.$$

Squeeze Theorem

Evaluate the limit

$$\lim_{x \to \pi} (x - \pi)^{2017} \sin\left(\frac{1}{x - \pi}\right).$$

Good theorem

Prove:

Theorem

Let f and g be functions defined near a, except possibly at a.
IF

1. $\lim_{x \to a} f(x) = 0$, and
2. g is bounded.
   (This means that there exists $M > 0$ such that for all $x \in \mathbb{R}$, $|g(x)| \leq M$, except perhaps when $x = a$.)

THEN $\lim_{x \to a} f(x)g(x) = 0$.

Next Class: Monday Oct 2

Watch videos 13, 14, 15 in Playlist 2.